The Boltzmann equation

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Abstract

The Boltzmann equation is a integro-differential equation which describes the dynamics of a rarefied gas. It is one of the fundamental equations of statistical physics, and despite a long history it remains far from fully understood. In this talk, we will give a basic overview of the Boltzmann equation. We will give a heuristic derivation from first physical principles, discuss the basic properties of solutions, and briefly describe some of the interesting mathematical problems surrounding it. Along the way, we will prove the second law of thermodynamics and learn what temperature is.

1 Derivation of the Boltzmann equation

These notes are based on the excellent survey articles [1] and [2], which are far more comprehensive and delve into much more material.

Consider a gas of N identical particles in $\mathbb{R}^3$, which we model temporarily as hard spheres of a small radius r. We assume no external forces are active. The motion of the particles is governed by classical mechanics, i.e. they are determined by Newton's laws of motion. Therefore each particle moves in a straight line at constant speed, until it collides with another particle; the post-collisional trajectories are determined by assuming that all collisions are perfectly elastic (preserving momentum and kinetic energy).

We will take N large, but we assume also that the gas is *rarefied*: the average distance a particle traverses between two consecutive collisions is non-negligible relative to a characteristic length. Intuitively, this means that the gas is dilute enough that binary interactions between particles are the dominant interactions (ternary and higher-order interactions can be neglected); quantitatively, this means $Nr^3 \ll 1$ and $Nr^2 \sim 1$. This is a major assumption: in particular, it implies that the standard equations of fluid dynamics, the Navier-Stokes and Euler equations,

do not apply. Our goal is to derive an equation that describes the dynamics of this ensemble of particles.

Each particle (say with label (i)) is described by 6 coordinates: 3 coordinates for the position $x^{(i)}$, and 3 for the momentum $v^{(i)}$. The equations of motion that govern the gas are a system of $6N$ fully coupled ODEs; as N is assumed large, this system of ODEs is completely impractical for any computational purposes. We therefore adopt a statistical approach to describing the particle configuration, by introducing the *one-particle distribution function* $P^{(1)}(t,x,v)$. $P^{(1)}(t,x,v)$ is the probability density of finding a fixed particle (say the particle labeled by (1)) at the phase-space point $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ at time $t \in \mathbb{R}$. It can also be interpreted as analogous to the physical density of particles in phase space. Of course, with N fixed, $P^{(1)}$ is a finite sum of delta masses; the hope is that in a scaling limit as $N \to \infty$, $P^{(1)}$ can be effectively described by a single measurable function, for which we have a closed evolution equation. This equation will be the *Boltzmann equation*, and our goal in this section is to derive it under fairly basic assumptions.

1.1 The basic setup

We first assume that particles never undergo collisions. Under this scenario, since there are no external forces, all particles move in a straight line at constant speed, with the speed and direction given by the particle's momentum. Therefore the dynamics are given by simple transport, and the governing equation for the distribution function is given by

$$\partial_t P^{(1)} + v \cdot \nabla_x P^{(1)} = 0.$$

However, obviously this equation needs to be adjusted in the presence of collisions. This adjustment takes the form of two additional terms,

$$\partial_t P^{(1)} + v \cdot \nabla_x P^{(1)} = G - L. \tag{1.1}$$

G and L denote the *gain* and *loss* terms respectively. $L(t,x,v)\ dxdvdt$ is the expected number of particles whose position and momenta are in the ranges $[x, x+dx]$ and $[v, v+dv]$ at time t, but leave these ranges at some time in the interval $[t, t+dt]$ due to a collision in this interval. $G(t,x,v)\ dxdvdt$ is the expected number of particles whose position and momenta are outside the ranges $[x, x+dx]$ and $[v, v+dv]$ at time t, but enter these ranges at some time in the interval $[t, t+dt]$. Therefore L gives the net loss of particles at a certain location in phase space, and G the net gain. Our goal is to understand the form of G and L; this is where the physics of the collisions will enter.

1.2 Elastic collisions of hard spheres

Let us consider a collision between two particles. Prior to the collision, we describe the first particle by its phase-space coordinates (x,v); the second we describe similarly by (x_*,v_*). After the collision, the first particle's coordinates will be denoted (x',v') and the second particle's coordinates will be denoted (x'_*,v'_*).

We assume that collisions are *localized* in spacetime. That is, a collision is an event that takes place at a single point in space, and the duration of the interaction is effectively instantaneous, so that the two particles do not influence each others' trajectories prior to and after the instant of collision.

We also assume that collisions are perfectly elastic: they preserve momentum and kinetic energy. This means that

$$\begin{aligned} v+v_* &= v'+v'_*, \\ |v|^2+|v_*|^2 &= |v'|^2+|v'_*|^2. \end{aligned} \tag{1.2}$$

Simple manipulations show that this implies that v,v_*,v',v'_* lie on a sphere centered at $\frac{v+v_*}{2}=\frac{v'+v'_*}{2}$ of radius $\frac{|v-v_*|}{2}=\frac{|v'-v'_*|}{2}$. In fact, the four vectors form the vertices of a rectangle. Therefore it is often convenient to write the post-collisional momenta v' and v'_* in the *σ-representation:*

$$\begin{aligned} v' &= \frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma, \\ v'_* &= \frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma, \end{aligned} \tag{1.3}$$

where $\sigma\in S^2$.

1.3 The loss and gain terms

We introduce the *two-particle distribution function* $P^{(2)}(t,x,x_*,v,v_*)$. This function is the probability density of finding, at time t, the first particle at the phase-space location (x,v) and a second fixed particle at (x_*,v_*).

With this we can now give a count of the loss term L. We consider particle (1), whose phase space distribution is given by $P^{(1)}$, and which is currently located in the range $[x,x+dx]$ and $[v,v+dv]$. By shifting our reference frame, we imagine particle (1) to be a sphere at rest; then in this reference frame, particle (i) has momentum v_i-v. To simplify the calculation of collisions, we can also imagine this particle to be a sphere of radius $2r$, by imagining all other particles to be point

particles. Then a particle with coordinates (x_*, v_*) collides with particle (1) if x_* lies on the sphere of radius $2r$ centered at x.

For a fixed particle (i), any collision with particle (1) will send particle (1) outside of the range $[x, x+dx]$, $[v, v+dv]$. Therefore to calculate L we simply need to count the expected number of collisions of particle (1) with the remaining $N-1$ particles. Since all particles are identical, we find that $L = (N-1)\ell$, where ℓ is the expected number of collisions with a single fixed particle, say particle (2).

So how many collisions with particle (1) can particle (2) contribute? Since we are fixing the phase-space coordinates of particle (1) to be (x, v), the phase-space distribution of particle (2) is simply the two-particle distribution $P^{(2)}$ conditioned on particle (1) having coordinates (x, v); this is precisely $P^{(2)}(t, x, x_*, v, v_*)$. At the moment of collision we have $x_* = x + 2rn$ where $n \in S^2$. Conditioning on this as well, the momentum distribution of particle (2) at the moment of collision is $P^{(2)}(t, x, x+2rn, v, v_*)$.

Suppose we count the collisions that occur in an infinitesimal area $(2r)^2 dn$ on the sphere of radius $2r$. If we definitely know that particle (2) has pre-collisional momentum v_*, then (using a change of inertial frame) the momentum of particle (2) relative to particle (1) is $v_* - v$. Under this assumption, particle (2) can collide with the area $(2r)^2 dn$ in the time interval $[t, t+dt]$ if and only if it lies in a (slanted) cylinder over the base $(2r)^2 dn$ of height $|(v_* - v) \cdot n| dt$. Therefore the number of collisions of particle (2) with particle (1), occurring on an area $dS = (2r)^2 dn$ on the sphere of radius $2r$, when the coordinates lie in the ranges $[x, x+dx]$, $[v, v+dv]$, $[x_*, x_* + dx_*]$, $[v_*, v + dv_*]$, and $[t, t+dt]$ is given by

$$P^{(2)}(t, x, x+2rn, v, v_*) |(v_* - v) \cdot n| (2r)^2 dn dx dv dv_* dt.$$

Integrating over all possible locations on the sphere (all possible collision locations) and all possible momenta of particle (2), we obtain

$$\ell \, dx dv dt = dx dv dt \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(2)}(t, x, x+2rn, v, v_*) (2r)^2 |(v_* - v) \cdot n| \, dn dv_*. \quad (1.4)$$

Here $\mathcal{B}^-$ is the hemisphere $(v_* - v) \cdot n < 0$, indicating that particles must be moving closer together before collision. Integrating, we therefore find that the loss term is given by

$$L = (N-1)(2r)^2 \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(2)}(t, x, x+2rn, v, v_*) |(v_* - v) \cdot n| \, dn dv_*. \quad (1.5)$$

The gain term is calculated similarly, but the integration is over $\mathcal{B}^+$, the hemisphere $(v_* - v) \cdot n > 0$, indicating that the particles are moving away from each

other after collision. Therefore the gain term takes the form

$$G = (N-1)(2r)^2 \int_{\mathbb{R}^3} \int_{\mathcal{B}^+} P^{(2)}(t,x,x+2rn,v,v_*)|(v_*-v)\cdot n|\, dndv_*,$$

We now assume, on physical grounds, that the distribution $P^{(2)}$ is continuous at a collision, that is,

$$P^{(2)}(t,x,x_*,v,v_*) = P^{(2)}(t,x,x_*,v',v'_*).$$

Inserting this into the gain term, and making the change of variable $n \mapsto -n$, the gain term takes the form

$$G = (N-1)(2r)^2 \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(2)}(t,x,x+2rn,v',v'_*)|(v_*-v)\cdot n|\, dndv_*, \qquad (1.6)$$

Recalling the rarefied gas assumptions $Nr^3 \ll 1$ and $Nr^2 \sim 1$, we now take a scaling limit as $N \to \infty$, $\sigma \to 0$, and $Nr^2 \to C < \infty$. Then absorbing constants, and assuming some continuity on $P^{(2)}$, we find that

$$L \approx C \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(2)}(t,x,x,v,v_*)|(v_*-v)\cdot n|\, dndv_*. \qquad (1.7)$$

and

$$G \approx C \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(2)}(t,x,x,v',v'_*)|(v_*-v)\cdot n|\, dndv_*, \qquad (1.8)$$

This scaling limit is known as the *Boltzmann-Grad limit*; it describes the asymptotic regimes in which the Boltzmann equation can be expected to accurately describe the dynamics.

1.4 The molecular chaos assumption

So far we are not much better than where we started; after all, the unknown function we seek is $P^{(1)}$, and we have written down the gain and loss terms in terms of another function $P^{(2)}$, which we also do not understand. The way past this is Boltzmann's main insight. Since the number of particles N is large, for any two fixed particles a collision between the two is a rare event. Instead of tracking individual collisions, we now consider collisions to be random events, consisting of an interaction between randomly chosen particles. We also assume that the positions and momenta of distinct pre-collision particles are uncorrelated. This can be written in the form

$$P^{(2)}(t,x,x_*,v,v_*) = P^{(1)}(t,x,v)P^{(1)}(t,x_*,v_*)$$

for pre-collision particles, i.e. $(v_* - v) \cdot n < 0$. This enables us to rewrite the loss term as

$$L = C \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(1)}(t,x,v) P^{(1)}(t,x,v_*) |(v_* - v) \cdot n| \, dn dv_*.$$

The assumption that pre-collision particles have uncorrelated phase-space coordinates (or specifically, momenta) does not directly imply a similar reduction for G. This is because G is written in terms of the post-collision two-particle momenta. However, because the density is assumed to be continuous at a collision, we can simply rewrite the post-collision density in terms of the pre-collision momenta, and then proceed as for the loss term. Under this assumption, the gain term takes the form

$$G = C \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} P^{(1)}(t,x,v') P^{(1)}(t,x,v'_*) |(v_* - v) \cdot n| \, dn dv_*.$$

The assumptions we have made are highly nontrivial. From the assumption that pre-collision momenta are uncorrelated, and continuity of the density at the time of collision, we have concluded that the post-collision momenta are also uncorrelated. Boltzmann in fact assumed this directly; this is the *molecular chaos hypothesis*.

Clearly, this is nonsense from a rigorous probabilistic standpoint: even if pre-collision momenta are uncorrelated, the moment two particles collide their post-collision momenta must be correlated. The molecular chaos assumption must therefore be interpreted as a statement that these correlations are negligible in the scaling limit. Through it, we in fact leave the realm of strictly classical mechanics, as we will see in the discussion of the H-theorem.

In any case, we now have the loss and gain terms in terms of $P^{(1)}$ only, so the system can finally be closed. Rescaling to eliminate the constant C, and using f to denote $P^{(1)}$ from here onward, we are at last able to write down the *Boltzmann equation for hard spheres*:

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^3} \int_{\mathcal{B}^-} [f' f'_* - f f_*] |(v_* - v) \cdot n| dn dv_*. \tag{1.9}$$

Here we adopt the standard notational convention

$$f = f(t,x,v), \; f_* = f(t,x_*,v_*), \; f' = f(t,x',v'), \; f'_* = f(t,x'_*,v'_*).$$

We derived this equation in the context of elastic collisions between hard spheres. However, in principle, we can perform the same analysis for other physical situations where the inter-particle interactions are of a different nature, such as charged

particles interacting under the Coulomb force. Therefore it is common to study the general *Boltzmann equation*

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^3} \int_{S^2} [f' f'_* - f f_*] B(n, |v - v_*|) dn dv_*. \tag{1.10}$$

The function $B = B(n, |v - v_*|)$ is called the *collision kernel*, and its form depends on the physics of the particle interactions under consideration. Two important examples of collision kernels are the hard-sphere collision kernel derived previously, and Coulomb interaction kernel

$$B(n, |v - v_*|) = \frac{1}{|v - v_*|^3 \sin^4(\theta/2)},$$

where θ is the angle between n and $v - v_*$.

2 Basic properties of the Boltzmann equation

We now study the basic properties of the Boltzmann equation (1.10). It is often written in the form

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

where

$$Q(f, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} [f' f'_* - f f_*] B(n, |v - v_*|) dn dv_* \tag{2.1}$$

is known as the *collision integral*.

2.1 Collision invariants

Recalling that f has the interpretation as a probability density or density of particles, it is natural to study quantities of the form

$$\int_{\mathbb{R}^3} f(t, x, v) \phi(v) \, dv,$$

as these have interpretations as expectations. Using the equation and differentiating under the integral sign, we obtain the expression

$$\partial_t \int_{\mathbb{R}^3} f \phi \, dv + v \cdot \int_{\mathbb{R}^3} \phi \nabla_x f \, dv = \int_{\mathbb{R}^3} Q(f, f) \phi \, dv.$$

Therefore it is important to understand integrals of the form $\int Q(f,f)\phi\ dv$. A basic property of the collision integral is that it is symmetric under the change of variable $v \mapsto v_*$, and antisymmetric under the change of variable $v \mapsto v', v_* \mapsto v'_*$. Using this, from a quick calculation it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3} Q(f,f)(v)\phi(v)\ dv \\ &\quad = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}[f'f'_* - ff_*][\phi+\phi_*-\phi'-\phi'_*]B(n,|v-v_*|)\ dndv_*dv.\end{aligned} \tag{2.2}$$

In particular, this expression vanishes whenever ϕ is a function satisfying

$$\phi + \phi_* - \phi' - \phi'_* = 0$$

everywhere in momentum space. Such functions are called *collision invariants*. As it turns out, the collection of all collision invariants is very easily understood: every collision invariant takes the form

$$\phi(v) = a + b\cdot v + c|v|^2,$$

where $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$ are constants. These are linear combinations of the *elementary collision invariants* 1, v, and $|v|^2$. The observables corresponding to them are given special names:

$$\begin{aligned} M &= \int_{\mathbb{R}^3} f(t,x,v)\ dv \quad \text{(mass)}, \\ P &= \int_{\mathbb{R}^3} vf(t,x,v)\ dv \quad \text{(momentum)}, \\ E &= \int_{\mathbb{R}^3} |v|^2 f(t,x,v)\ dv \quad \text{(energy)}. \end{aligned} \tag{2.3}$$

2.2 Boltzmann's H-functional

Next, we take $\phi = \log f$. In this case

$$\phi + \phi_* - \phi' - \phi'_* = \log(ff_*) - \log(f'f'_*).$$

From the elementary inequality

$$[f'f'_* - ff_*][\log(ff_*) - \log(f'f'_*)] \le 0,$$

we conclude the *Boltzmann inequality*:

$$\int_{\mathbb{R}^3} Q(f,f)\log(f)\,dv \leq 0.$$

The corresponding observable

$$H = \int_{\mathbb{R}^3} f\log f\,dv$$

is called *Boltzmann's H-functional*, or the *entropy*.

2.3 Maxwellian distributions

If equality holds in the Boltzmann inequality, then $\log f$ is a collision invariant, i.e.

$$f(t,x,v) = \exp(a + b\cdot v + c|v|^2).$$

Such a distribution, with $c < 0$ (so that f is integrable) is called a *Maxwellian distribution*. Conversely, if f is a Maxwellian distribution, then equality holds in the Boltzmann inequality. Moreover, every Maxwellian distribution is a solution to (1.10), because if $f = \exp(\phi)$ where ϕ is a collision invariant, then $f'f'_* - ff_* = 0$. Therefore $Q(f,f) = 0$, and $\nabla_x f = 0$ since f is independent of x. Therefore Maxwellian distributions characterize those solution to (1.10) for which the Boltzmann inequality is an equality.

3 The spatially homogeneous equation

The full Boltzmann equation is a subject for which much remains to be studied, and which we cannot hope to cover in the span of these notes alone. We therefore restrict to a simpler but still highly nontrivial case, that in which f is constant in the x variable: that is, $f(t,x,v) = f(t,v)$. This describes a gas which is equidistributed in space; then the goal is to understand the momentum distribution of the gas, and its relation to physical quantities. Such a distribution is said to be *spatially homogeneous*, and satisfies the *spatially homogeneous Boltzmann equation*

$$\partial_t f = Q(f,f). \tag{3.1}$$

With the transport term removed from the equation, the collision invariants of the previous section become conserved quantities:

$$\partial_t M = 0,\ \partial_t P = 0,\ \partial_t E = 0. \tag{3.2}$$

Moreover, the Boltzmann inequality becomes an expression of the *second law of thermodynamics*:

$$\partial_t H \leq 0. \tag{3.3}$$

This inequality is a special case of the *Boltzmann H-theorem*; the general H-theorem is the analogous statement for the spatially inhomogeneous equation. This last inequality marks an important difference between the classical description of the gas as a many-particle system of Newtonian particles and the Boltzmann description. One fundamental property of the classical description is that it is *time-reversible*: given the positions and momenta of all particles at a given time, one can (in principle) determine the dynamics not only forward in time, but also backward (by reversing all momenta, and then running this new system forward in time). In other words, time has no preferred direction: the equations of motion are symmetric under time reversal.

In the Boltzmann description this is explicitly violated. Because the entropy is nonincreasing in forward time, there is clearly a distinction between different directions of time; the Boltzmann equation is *time-irreversible*. The apparent discrepancy between time-reversible microscopic dynamics and time-irreversible macroscopic dynamics is a subject of much interesting physical debate, but unfortunately it is beyond the scope of these notes.

Maxwellian distributions $f = A\exp(-\beta|v|^2)$ are solutions to the homogeneous equation with maximal entropy. A spatially homogeneous gas whose momentum distribution is Maxwellian is called an *ideal gas*. Ideal gases are standard models for the behavior of gases in the sciences. For an ideal gas, notions such as temperature are well-defined, and appear in the constants defining the corresponding Maxwellian distribution. That is, each Maxwellian distribution (with zero drift) can be written in the form

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{m|v|^2}{2kT}\right),$$

where m is the mass of particles, k is a physical constant known as the *Boltzmann constant*, and T is the *thermodynamic temperature*. (We are enforcing that f is a probability density, so that it is normalized to have unit integral.)

4 Interesting problems on the Boltzmann equation

The Boltzmann equation has been a subject of intense study ever since it was first formulated. However, much remains to be done for the mathematical theory

of the equation. A major reason is that the collision integral $Q(f,f)$ is fairly complicated. Another reason is that the Boltzmann equation, through the choice of the collision kernel B, is quite general, and there is very little that can be said that holds in general for all collision kernels. Therefore each collision kernel, corresponding to different physical scenarios, must be studied essentially on its own. In this section we will briefly discuss some of the mathematical problems surrounding the Boltzmann equation, many of which remain open and active areas of research.

4.1 Well-posedness

From a PDE perspective, the first question of interest is well-posedness: existence and uniqueness of solutions, and continuous dependence on initial data. This is a famously difficult problem, even for the spatially homogeneous equation. One of the major difficulties is that, aside from the conservation laws and the H-functional, there are no known generally valid globally controlled quantities for the evolution. Also, the equation itself is sometimes difficult to interpret: for hard-sphere collisions things are generally fine, but for collision kernels B describing other particle interactions it can be hard to justify the finiteness of the collision integral $Q(f,f)$. This is especially true with long-range forces, which result in collision kernels that exhibit a singularity along certain collision angles.

Much of the progress on well-posedness is relatively recent. For the large part it is limited to the spatially homogeneous case with a certain class of collision kernels which behave similarly to hard-sphere collisions. There has been more success for well-posedness of solutions which are perturbations of a Maxwellian distribution, for which one can linearize the collision integral around the Maxwellian.

4.2 Boundary conditions

We have derived and stated properties for the Boltzmann equation assuming that the gas is distributed over all of $\mathbb{R}^3$; the only boundary conditions we imposed were integrability. However, for physical applications it is also important to consider gases in bounded domains. For the most part everything we have said so far applies in bounded domains. The equations remain largely the same, the collision invariants and H-functional have their usual properties, and Maxwellian distributions are still entropy-maximal solutions.

However, introducing the appropriate boundary conditions is a nontrivial task. Beyond complicating the well-posedness problem, the most mathematically natu-

ral boundary conditions are not necessarily the best suited to describing physical scenarios. Mathematically, boundary conditions such as *specular reflection* (particles bounce off the walls at an angle equal to the pre-collision angle) and *bounce-back* (particles bounce off the walls with reversed momenta) are convenient, but they are not always good physical models. A good boundary condition takes into account the fine details of the gas-surface interaction. A commonly studied example of such a boundary condition is *Maxwellian diffusion*, in which particles are absorbed and re-emitted by the boundary according to a probability distribution which maintains thermodynamic equilibrium between the particles and the boundary.

4.3 Trend to equilibrium

The general conjecture for the long-time behavior of solutions to (1.10) in bounded domains is that they should asymptotically approach a Maxwellian distribution. That is, gases in bounded domains should eventually behave as ideal gases, a conjecture suggested by the H-theorem. The problem of trend to equilibrium is to prove this assertion, along with a quantitative description of the rate of convergence. Progress on this problem was driven by the development of entropy dissipation estimates, and nowadays has a mostly satisfactory answer when the corresponding Cauchy problem has a good theory.

In unbounded domains, the conjecture for the long-time behavior of solutions to the spatially inhomogeneous equation is different. This is because in unbounded domains, the long-time effects of the transport operator become important. The idea is that the transport operator has a *dispersive* effect on the distribution: some particles may fail to undergo a large number of collisions, and instead disperse out to infinity. For this situation the conjecture is that particles are dispersed at infinity, and the problem is to find quantitative estimates on the rate of dispersion. The research in this direction is driven by Strichartz estimates for the transport operator, which bears similarities to the theory of the Schrödinger equation.

4.4 Hydrodynamic limit

The Boltzmann equation is an equation intermediate between classical many-particle mechanics and continuum mechanics. In particular, in some sense it lies between the equations of motion for classical mechanics and the fundamental

equations of fluid dynamics: the Navier-Stokes and Euler equations. The problem of the hydrodynamic limit is to demonstrate that the Navier-Stokes and Euler equations arise as scaling limits of the Boltzmann equation, where the scaling limit is along the passage from a dilute gas to a fluid. This can be done at the level of purely formal manipulations, but to do so rigorously is an open problem. This problem is of interest partly for its physical implications, but also because understanding the passage between Boltzmann and fluid equations can be sources of inspiration in terms of what theorems can be passed between them.

4.5 Variant equations

The Boltzmann equation as stated in (1.10) describes the dynamics of a dilute monatomic gas, and the particle interactions are left fairly vague in the form of the collision kernel B. Extensions and variants of the Boltzmann equation exist and are used to describe more complicated physical situations. We describe two of them here, though there are many more that are studied.

4.5.1 Mixtures and polyatomic gases

In a monatomic gas, one can assume that all particle interactions are identical. For mixtures, one has to take into account that the interaction between two particles depends on which type of particles are interacting. Equations for such systems can be derived analogously to how we derived the Boltzmann equation for monatomic gases. The main difference is that we now need N distribution functions f_i, one for each species in the gas, and we need to study the gain and loss terms for each one. The gain and loss terms for f_i split into gain and loss terms for f_i due to interactions with each species. Naturally, being a system with a more complicated nonlinearity, the Boltzmann equations for mixtures are much less well-studied analytically, though they are useful for modelling. One useful feature of this approach is that one can model chemical reactions and radiative processes happening inside the gas as well, by incorporating them into the gain and loss terms.

4.5.2 Quantum Boltzmann equations

When one incorporates quantum effects into the particle interactions, very different effects can appear.

For instance, one might consider a gas of fermions. Fermions satisfy the Pauli exclusion principle; in particular, they prefer to be in different quantum states when possible. That is, particles find it easier to transfer from momentum state v to state v' if $f(v')$ is low, which results in an anti-clustering effect. The equation corresponding to a gas of fermions is called the *Boltzmann-Fermi* equation. It is identical to the spatially homogeneous Boltzmann equation, except that the collision integral gains a cubic term:

$$\begin{aligned} &Q(f,f)(v) \\ &= \int_{\mathbb{R}^3}\int_{S^2}[f'f'_*(1+\varepsilon f)(1+\varepsilon f_*) - ff_*(1+\varepsilon f')(1+\varepsilon f'_*)]B(n,|v-v_*|)dndv_* \end{aligned} \tag{4.1}$$

with $\varepsilon < 0$. Bosons do not satisfy the Pauli exclusion principle, and therefore they tend to cluster in phase space. That is, particles find it easier to transfer from momentum state v to state v' if $f(v')$ is high. They satisfy the *Boltzmann-Bose* equation, for which the collision integral is the same as (4.1) but with $\varepsilon > 0$. This equation is particularly well-studied as one of several models of Bose-Einstein condensation.

All of the same questions for Boltzmann equations can be asked for quantum Boltzmann equations. They have collision invariants, an H-functional, and equilibrium solutions; the questions of rigorous derivation, well-posedness, and asymptotic behavior are of significant interest. This area has seen much very recent progress, both on the mathematical side and on the level of physical experiments, as it has become possible to do experiments with ultracold atoms.

References

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