A transference principle in fractional calculus

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In this short note we are concerned with the following statements regarding fractional calculus on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$:

Proposition 1 (fractional product rule)**.** *Let $d \geq 1$, $s > 0$, $1 < p < \infty$, and $1 < p_2, q_2 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\||\nabla|^s(fg)\|_{L^p(\mathbb{T}^d)} \lesssim \||\nabla|^s f\|_{L^{p_1}(\mathbb{T}^d)} \|g\|_{L^{p_2}(\mathbb{T}^d)} + \||\nabla|^s g\|_{L^{q_1}(\mathbb{T}^d)} \|f\|_{L^{q_2}(\mathbb{T}^d)}.$$

Proposition 2 (nonlinear Bernstein)**.** *Let $G : \mathbb{C} \to \mathbb{C}$ be Hölder continuous of order $\alpha \in (0,1]$. Let $d \geq 1$ and $1 \leq p \leq \infty$. Then for $u : \mathbb{T}^d \to \mathbb{C}$ smooth and periodic, we have*

$$\|P_N G(u)\|_{L^{p/\alpha}(\mathbb{T}^d)} \lesssim N^{-\alpha} \|\nabla u\|^{\alpha}_{L^p(\mathbb{T}^d)}$$

for all $N > 1$.

Proposition 3 (fractional chain rule)**.** *Suppose $F : \mathbb{C} \to \mathbb{C}$ satisfies $|F(u) - F(v)| \lesssim |u - v|(G(u) + G(v))$ for some $G : \mathbb{C} \to [0,\infty)$. Let $d \geq 1$, $0 < s < 1$, $1 < p < \infty$, and $1 < p_2 \leq \infty$, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\||\nabla|^s F(u)\|_{L^p(\mathbb{T}^d)} \lesssim \||\nabla|^s u\|_{L^{p_1}(\mathbb{T}^d)} \|G(u)\|_{L^{p_2}(\mathbb{T}^d)}.$$

Propositions 1 and 3 are well-known. They were first proved in the Euclidean setting (replacing $\mathbb{T}^d$ by $\mathbb{R}^d$) by Christ and Weinstein [1]. A textbook treatment of these results can be found in Chapter 2 of [3]. A proof of Proposition 2 in Euclidean space can be found, for instance, in [2].

While it is generally accepted that the above statements are true on $\mathbb{T}^d$, a rigorous proof of this assertion can be difficult to locate in the literature. It is common that the Euclidean version of the statement is cited via [1], and then the periodic version is justified by appealing to a transference principle without further elaboration. It is the purpose of this note to make the transference principle rigorous, at least in the context of Propositions 1, 2, and 3.

We first note that Proposition 1 has identical proofs in the Euclidean and periodic settings. If one examines the proof in the Euclidean setting, it only uses basic facts about maximal functions, the Littlewood-Paley square-function estimate, and the Fefferman-Stein vector-valued maximal inequality. All of these results remain valid in the periodic setting.

Propositions 2 and 3 require a little more work. For the Euclidean case, the proof (at least, the proof that the author is familiar with) uses some facts specific to Euclidean Littlewood-Paley theory. In particular, the Euclidean Littlewood-Paley convolution kernels are Schwartz functions, and have a dilation structure $\psi_N^\vee(x) = N^d \psi^\vee(Nx)$, with associated estimates.

For Littlewood-Paley theory on the torus, neither property makes sense: there is no sensible notion of a Schwartz function on the torus, and there is no dilation structure because the torus is not dilation-invariant. However, there is an estimate on the periodic Littlewood-Paley convolution kernels that effectively does the same job. We will state and prove this estimate, and use this to prove Proposition 2 to illustrate its usage.

Lemma 4. *Let $d \geq 1$. Let $m = (m_1, \dots, m_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ be multi-indices of non-negative integers. Let ψ_N denote the Fourier multiplier defining the Littlewood-Paley projection P_N. Then for $y = (y_1, \dots, y_d) \in \mathbb{T}^d = [0, 2\pi)^d$,*

$$|D^\beta (\psi_N)^\vee(y)| \lesssim_{m,\beta} N^{d+|\beta|} \prod_{j=1}^{d} \frac{1}{(1+N\tilde{y}_j)^{m_j}},$$

where $|\beta| = \sum_1^d \beta_j$, $D^\beta = \partial_{y_1}^{\beta_1} \cdots \partial_{y_d}^{\beta_d}$, and

$$\tilde{t} = \begin{cases} t & 0 \leq t \leq \pi, \\ 2\pi - t & \pi \leq t < 2\pi. \end{cases}$$

Lemma 4 states that $\psi_N^\vee$ behaves like a dilation to scale $\frac{1}{N}$ of a function that obeys Schwartz-type decay, so long as we stay away from 0 and 2π. Of course, at 0 and 2π there can be no Schwartz-type decay, as here there is no cancellation in the complex exponentials.

Proof. First take $d = 1$. For $N \geq 1$, we write

$$\psi_N^\vee(y) = \sum_{n=-\infty}^{\infty} \psi_N(n) e^{iny} = \sum_{n=-\infty}^{\infty} \psi_1(\frac{n}{N}) e^{iny},$$

and thus

$$\frac{d^\beta}{dy^\beta}\psi_N^\vee(y) = \sum_{n=-\infty}^{\infty}(in)^\beta \psi_1(\frac{n}{N})e^{iny} = (iN)^\beta \sum_{n=-\infty}^{\infty}\left(\frac{n}{N}\right)^\beta \psi_1(\frac{n}{N})e^{iny}$$
$$= (iN)^\beta \sum_{n=-\infty}^{\infty}\Psi_\beta(\frac{n}{N})e^{iny},$$

where $\Psi_\beta(t) = t^\beta \psi_1(t)$. Summing by parts,

$$\left|\sum_{n=-\infty}^{\infty}\Psi_\beta(\frac{n}{N})e^{iny}\right| = \frac{1}{|1-e^{iy}|}\left|\sum_{|n|\sim N} e^{iny}[\Psi_\beta(\frac{n}{N}) - \Psi_\beta(\frac{n-1}{N})]\right|$$
$$\lesssim \|\frac{d}{dx}\Psi_\beta\|_{L^\infty(\mathbb{R})}\frac{N^{-1}}{|1-e^{iy}|}\sum_{|n|\sim N} 1$$
$$\sim \|\frac{d}{dx}\Psi_\beta\|_{L^\infty(\mathbb{R})}\frac{N^{1-1}}{|1-e^{iy}|}.$$

Similarly, if we sum by parts m times, we end up with the estimate

$$\left|\sum_{n=-\infty}^{\infty}\Psi_\beta(\frac{n}{N})e^{iny}\right| \lesssim \|\frac{d^m}{dx^m}\Psi_\beta\|_{L^\infty(\mathbb{R})}\frac{N^{1-m}}{|1-e^{iy}|^m} \sim_{m,\beta} \frac{N^{1-m}}{|1-e^{iy}|^m}.$$

Now we seek to strike a balance between the estimate

$$\left|\frac{d^\beta}{dy^\beta}\psi_N^\vee(y)\right| \lesssim_{m,\beta} \frac{N^{1+\beta-m}}{|1-e^{iy}|^m},$$

which is singular at 0 and 2π, and the trivial estimate

$$\left|\frac{d^\beta}{dy^\beta}\psi_N^\vee(y)\right| \lesssim N^{1+\beta}.$$

We break into cases:

1. $0 \le y < \frac{1}{N}$: In this range, by the trivial estimate and $1 \sim 1+Ny$ we obtain

$$\left|\frac{d^\beta}{dy^\beta}\psi_N^\vee(y)\right| \lesssim N^{1+\beta} \sim \frac{N^{1+\beta}}{(1+Ny)^m}.$$

2. $\frac{1}{N} \leq y < \frac{\pi}{2}$: In this range, $|1-e^{iy}| \sim |\sin y| \sim y$, and $Ny \gtrsim 1+Ny$, so

$$\left|\frac{d^\beta}{dy^\beta}\psi_N^\vee(y)\right| \lesssim \frac{N^{1+\beta}}{(Ny)^m} \lesssim \frac{N^{1+\beta}}{(1+Ny)^m}.$$

3. $\frac{\pi}{2} \leq y < \frac{3\pi}{2}$: In this range, $N|1-e^{iy}| \sim N \sim (1+Ny)$, so

$$\left|\frac{d^\beta}{dy^\beta}\psi_N^\vee(y)\right| \lesssim \frac{N^{1+\beta}}{N^m|1-e^{iy}|^m} \sim \frac{N^{1+\beta}}{(1+Ny)^m}.$$

4. $\frac{3\pi}{2} \leq y < 2\pi - \frac{1}{N}$: In this range $|1-e^{iy}| \sim |\sin(2\pi - y)| \sim 2\pi - y$, and we argue as in 2.

5. $2\pi - \frac{1}{N} \leq y < 2\pi$: We argue as in 1.

Thus the claim is proved for $d = 1$. The case of higher dimensions is similar: write

$$D^\beta \psi_N^\vee(y) = (iN)^{|\beta|}\sum_{\xi_d} e(\xi_d y_d)\cdots\sum_{\xi_1} e(\xi_1 y_1)\Psi_\beta(\frac{\xi}{N}),$$

where $\Psi_\beta(t_1,\ldots,t_d) = t_1^{\beta_1}\cdots t_d^{\beta_d}\psi_1(t_1,\ldots,t_d)$. Applying the summation by parts argument successively in each variable produces

$$|D^\beta \psi_N^\vee(y)| \lesssim_{\beta,m} N^{|\beta|}\prod_{j=1}^{d}\frac{N^{1-m_j}}{|1-e(y_j)|^{m_j}},$$

where the implicit constant is essentially $\|D^m\Psi_\beta\|_{L^\infty(\mathbb{R}^d}$. From here we estimate each factor as in the one-dimensional case, and the claim follows. □

The virtue of Lemma 4 is that it can be used in the periodic setting to replace arguments in Euclidean Littlewood-Paley theory that invoke the dilation structure. As an example, we apply it to the proof of Proposition 2.

Proof of Proposition 2. Let us take $p < \infty$; the changes required for $p = \infty$ will be obvious. First take $d = 1$. We observe that for $N > 1$, $\psi_N^\vee$ is a mean-zero function on $\mathbb{T}$. Therefore we have the pointwise estimate

$$
\begin{aligned}
|P_N G(u)(x)| &= |(\psi_N^\vee * G(u))(x)| \\
&= \left| \int_{\mathbb{T}} \psi_N^\vee(y)[G(u(x-y)) - G(u(x))] \, dy \right| \\
&\leq \left| \int_{\mathbb{T}} \psi_N^\vee(y) 1_{[0,\pi)}(y)[G(u(x-y)) - G(u(x))] \, dy \right| \\
&+ \left| \int_{\mathbb{T}} \psi_N^\vee(y) 1_{[\pi,2\pi)}(y)[G(u(x-y)) - G(u(x-2\pi))] \, dy \right| \\
&\leq \int_{\mathbb{T}} |\psi_N^\vee(y) 1_{[0,\pi)} \left(\int_0^1 |y||\nabla u(x-\theta y)| \, d\theta \right)^\alpha dy \\
&+ \int_{\mathbb{T}} |\psi_N^\vee(y) 1_{[\pi,2\pi)}(y) \left(\int_0^1 |2\pi - y||\nabla u(x - 2\pi + \theta(2\pi - y))| \, d\theta \right)^\alpha dy,
\end{aligned}
$$

where we have used the periodicity and Hölder continuity of u. Taking the $L^{p/\alpha}$ norm and applying Minkowski's inequality, we obtain

$$
\begin{aligned}
\|P_N G(u)\|_{L_x^{p/\alpha}(\mathbb{T}^d)} \lesssim{} & \|\nabla u\|_{L^p(\mathbb{T})}^\alpha \int_{\mathbb{T}} |y|^\alpha 1_{[0,\pi)}(y) |\psi_N(y)| \, dy \\
&+ \|\nabla u\|_{L^p(\mathbb{T})}^\alpha \int_{\mathbb{T}} |2\pi - y|^\alpha 1_{[\pi,2\pi)}(y) |\psi_N(y)| \, dy.
\end{aligned}
$$

It now suffices to prove

$$
\int_{\mathbb{T}} |y|^\alpha 1_{[0,\pi)}(y)|\psi_N(y)| \, dy + \int_{\mathbb{T}} |2\pi - y|^\alpha 1_{[\pi,2\pi)}(y)|\psi_N(y)| \, dy \lesssim N^{-\alpha}.
$$

Up to now the proof is essentially identical to the Euclidean case; at this point in the Euclidean setting one expresses ψ_N as a dilate of ψ_1 and uses the Schwartz decay. Instead we apply Lemma 4 and change variables, obtaining:

$$
\begin{aligned}
\int_{\mathbb{T}} |y|^\alpha 1_{[0,\pi)}(y)|\psi_N(y)| \, dy &\lesssim_m \int_{\mathbb{T}} |y|^\alpha \frac{N}{(1+Ny)^m} \, dy \\
&= N^{-\alpha} \int_{N\mathbb{T}} \frac{|x|^\alpha}{(1+x)^m} \, dy \\
&\leq N^{-\alpha} \int_{\mathbb{R}} \frac{|x|^\alpha}{(1+|x|)^m} \, dy \\
&\lesssim N^{-\alpha},
\end{aligned}
$$

so long as we choose $m > 1+\alpha$. The other integral is estimated similarly. This establishes the claim for $d = 1$. For $d \geq 2$, the argument is similar. The only change is that instead of $\mathbb{T} = [0,\pi) \cup [\pi, 2\pi)$, we perform a binary decomposition of $\mathbb{T}^d$ into 2^d equal-sized cubes; e.g. $\mathbb{T}^2$ is the union of $[0,\pi)^2$, $[0,\pi) \times [\pi, 2\pi)$, $[\pi, 2\pi) \times [0,\pi)$, and $[\pi, 2\pi)^2$. We break up the convolution integral correspondingly, and use Lemma 4 to estimate each piece. The details are left to the reader. □

A similar application of Lemma 4 can be used to transfer the proof of Proposition 3 from the Euclidean setting to the periodic setting; the details are left to the reader.

References

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[2] R. Killip and M. Vişan, *Nonlinear Schrödinger equations at critical regularity*, 2013. `https://www.math.ucla.edu/~visan/ClayLectureNotes.pdf`.

[3] M. Taylor, *Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, American Mathematical Society, 2000.