Selected geometry & topology qualifying exam solutions

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These are solutions to some (not all) UCLA geometry/topology qualifying exam problems. More recent exams are better represented.

The primary references used include:

Introduction to Smooth Manifolds, 2nd ed., by John M. Lee.

Geometry of Differential Forms, by Shigeyuki Morita.

Differential Topology, by Victor Guillemin and Alan Pollack.

Algebraic Topology, by Allen Hatcher.

Algebraic Topology: A First Course, by William Fulton.

Ian Coley's qualifying exam solutions.

Austin Christian's solutions for Fall 2016.

1 Navigation

Click on the following links to go to different exams.

Spring 2012
Fall 2011
Fall 2012
Spring 2013
Fall 2013
Spring 2014
Fall 2014
Spring 2015
Fall 2015
Spring 2016
Fall 2016

2 Winter 2002

Problem. (W02:05)

Consider the 2-form on $\mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\sigma = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy).$$

(a) Show that σ is closed in $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

(b) Show that the 2-form

$$\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$$

is closed but not exact on S^2.

(c) Find $\int_{S^2} \omega$.

(d) Suppose M is compact (closed?), 2-dimensional, oriented embedded submanifold of $\mathbb{R}^3 \setminus \{(0,0,0)\}$. What are the possible values of $\int_M \sigma$?

Solution.

(a) Routine calculation.

(b) Closed: all 2-forms are closed on S^2. Exact: regard ω as the restriction of the same form on B^3, the closed unit ball in $\mathbb{R}^3$, with $\partial B^3 = S^2$. Then by Stokes' theorem,

$$\int_{S^2} \omega = \int_{B^3} d\omega = \int_{B^3} 3dx \wedge dy \wedge dz = 4\pi.$$

In particular, ω cannot be exact, because an exact form integrates to 0 over S^2.

(c) See part (b).

(d) Presumably this has something to do with Poincaré duality, but I'm not sure how exactly.

3 Spring 2002

Problem. (S02:06)

(a) Show that if $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.

(b) Show that if X has S^{2n} as its universal cover, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}/2\mathbb{Z}$.

(c) Show that if X has S^{2n+1} as its universal cover, then X is orientable.

Solution.

(a) We claim that f is homotopic to the antipodal map $x \mapsto -x$, which has degree $(-1)^{n+1}$. Indeed, the straight-line homotopy

$$F_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

is well-defined, because if $x \in S^n$, then $|x| = |f(x)| = 1$; if $|(1-t)f(x) - tx| = 0$, then necessarily

$$1 - t = |(1-t)f(x)| = |tx| = t,$$

so $t = \frac{1}{2}$, but then we would have $\frac{1}{2}f(x) = \frac{1}{2}x$, which is impossible because f fixes no points.

(b) Suppose S^{2n} covers X. Then $X = S^{2n}/G$, where G is the group of deck transformations. We observe that since S^{2n} is path connected, each non-identity deck transformation g is a homeomorphism $S^{2n} \to S^{2n}$ that fixes no points. Then $\deg g = (-1)^{2n+1} = -1$. But then it follows that G is either trivial, or $\mathbb{Z}/2\mathbb{Z}$: for suppose G contains two nontrivial elements f and g. Then $\deg(fg) = \deg(f)\deg(g) = 1$. Then by our observation, it must be that $fg = \mathrm{id}_{S^{2n}}$, and so $f = g^{-1}$, and $g^2 = \mathrm{id}_{S^{2n}}$, so $f = g$. Now, since S^{2n} is simply connected, we have

$$\pi_1(X) \cong G,$$

and thus $\pi_1(X)$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$.

(c) If S^{2n+1} covers X, then every nontrivial deck transformation has degree $(-1)^{2n+2} = 1$. By the Hopf degree theorem, every deck transformation is homotopic to the identity. In particular every deck transformation is locally orientation-preserving. Therefore the orientation on S^{2n+1} and the quotient map $S^{2n+1} \to S^{2n+1}/G = X$ induces an orientation on X.

4 Fall 2003

Problem. (F03:09) If a closed manifold M has S^{2n+1} as a covering space, $n \geq 1$, then M is orientable.

Solution. See S02:06.

5 Fall 2004

Problem. (F04:07) Let $X_1 = S^1 \vee S^2$ and $X_2 = S^1 \vee S^1$.

(a) Find $\pi_1(X_1)$ and $\pi_1(X_2)$.

(b) Find their universal coverings.

Solution.

(a) $\pi_1(X_1) = \pi_1(S^1) * \pi_1(S^2) = \mathbb{Z} * 0 = \mathbb{Z}$. $\pi_1(X_2) = \mathbb{Z} * \mathbb{Z}$.

(b) The universal cover of X_1 is the following space:

$$\mathbb{R} \coprod \left(\coprod_{n\in\mathbb{Z}} (S^2)_n\right) / \sim,$$

where for each n we identify a point $x_n \in (S^2)_n$ with $n \in \mathbb{R}$. (Essentially a string of S^2-s attached at the integers.) The covering map is given by the usual coverings $\mathbb{R} \to S^1$, and the identity on S^2. The universal cover of X_2 is the Cayley graph of the free group on two generators.

6 Winter 2005

Problem. (W05:10)

(a) Find the Euler characteristic of X_4^2, the 2-skeleton of the 4-simplex.

(b) Give a reason why $H_2(X_4^2)$ is free abelian and find its rank.

Solution.

(a) By the cellular definition of Euler characteristic, we can count the number of k-cells in X_4^2, where $k = 0,1,2$, and take the alternating sum. Regarding X_4 as $[v_0, \ldots, v_4]$, each k-cell is determined uniquely by choosing k of the vertices in any order. Therefore the number of k-cells is $\binom{5}{k}$. So

$$\chi(X_4^2) = \binom{5}{0} - \binom{5}{1} + \binom{5}{2} = 6.$$

(b) The long exact sequences for the pairs (X_4^2, X_4^1) and (X_4^1, X_4^0) give us the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & H_2(X_4^2) \arrow[r] & H_2(X_4^2, X_4^1) \arrow[r] & H_1(X_4^1) \arrow[r] \arrow[d] & 0 \\
 & & & H_1(X_4^1, X_4^0) \arrow[d] & \\
 & & & H_0(X_4^0) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

where the vertical and horizontal sequences are exact. $H_2(X_4^2, X_4^1)$ and $H_1(X_4^1, X_4^0)$ are the second and first cellular chain groups, and hence are free abelian with bases in correspondence with the 2-cells and 1-cells: therefore those are $\mathbb{Z}^6$ and $\mathbb{Z}^5$ respectively. X_4^0 consists of 5 0-cells, so $H_0(X_4^0) = \mathbb{Z}^5$. Therefore substituting the information about our homology groups, the diagram is

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & H_2(X_4^2) \arrow[r] & \mathbb{Z}^6 \arrow[r] & H_1(X_4^1) \arrow[r] \arrow[d] & 0 \\
 & & & \mathbb{Z}^5 \arrow[d] & \\
 & & & \mathbb{Z}^5 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

By exactness of the vertical sequence, we find that $H_1(X_4^1) \cong 0$, and therefore by exactness of the horizontal sequence $H_2(X_4^2) \cong \mathbb{Z}^6$.

7 Spring 2006

Problem. (S06:10) Let S^p and S^q be spheres of dimension $p, q \geq 0$. Compute the homology groups $H_n(S^p \times S^q)$ for all $n \geq 0$.

Solution. We assign to S^n the standard CW structure: one 0-cell v, and one n-cell e^n attached to v along the boundary. Denote the 0-cells of S^p and S^q by v and w respectively, and the 1-cells by e^p, e^q. Then the product CW structure on $S^p \times S^q$ consists of the following cells:

1. One 0-cell (v, w).

2. One p-cell (e^p, w).

3. One q-cell (v, e^q). (We could have $p = q$ as well.)

4. One $(p+q)$-cell (e^p, e^q).

The cellular boundary maps for S^n are $\partial_0 v = 0$, $\partial_n e^n = 0$: for $n \neq 1$ this is because there are no $n-1$-cells in the CW structure for S^n, and for $n = 1$ $\partial_1 e^1 = v - v = 0$. For the product, we therefore have the following cellular boundary maps:

$$\begin{aligned}
\partial_0(v,w) &= (\partial v, w) + (-1)^0 (v, \partial w) = 0; \\
\partial_p(e^p, w) &= (\partial_p e^p, w) + (-1)^p (e^p, \partial_0 w) = 0; \\
\partial_q(v, e^q) &= (\partial_0 v, e^q) + (-1)^0 (v, \partial_q e^q) = 0; \\
\partial_{p+q}(e^p, e^q) &= (\partial_p e^p, e^q) + (-1)^p (e^p, \partial_q e^q) = 0.
\end{aligned}$$

Therefore in all cases, im $\partial = 0$ and $\ker \partial$ is the whole group. Therefore homology is isomorphic to the corresponding cellular chain complex, and it remains to determine what those are.

(a) $p = 0$, $q \neq 0$. The cellular chain complex is

$$0 \longrightarrow \underset{q}{\mathbb{Z}^2} \longrightarrow \cdots \longrightarrow \underset{0}{\mathbb{Z}^2} \longrightarrow 0$$

(b) $p = 0, q = 0$: The cellular chain complex is

$$0 \longrightarrow \underset{0}{\mathbb{Z}^4} \longrightarrow 0$$

(c) $p, q \neq 0$, $p > q$: The cellular chain complex is

$$0 \longrightarrow \underset{p+q}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{p}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{q}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{0}{\mathbb{Z}} \longrightarrow 0$$

(d) $p, q \neq 0$, $p = q$: The cellular chain complex is

$$0 \longrightarrow \underset{2p}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{p}{\mathbb{Z}^2} \longrightarrow \cdots \longrightarrow \underset{0}{\mathbb{Z}} \longrightarrow 0$$

8 Spring 2009

Problem. (S09:12) Let $f : T \to T = S^1 \times S^1$ be a map of the torus inducing $f_\pi : \pi_1(T) \to \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, and let F be a matrix representing f_π. Prove that the determinant of F equals the degree of the map f.

Solution. Taking abelianizations, we note that the induced homomorphism on first homology

$$f_* : H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

is represented by precisely the same matrix F. Since the homology of T is free and finitely generated, by universal coefficients we may dualize in all dimensions to obtain maps on the integer cohomology ring

$$f^* : H^*(T) \to H^*(T),$$

and the map $f^* : H^1(T) \to H^1(T)$ is represented by the transpose F^* of F. Suppose now that x, y are generators of $H^1(T)$ for which F^* is given by

$$F^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $x^2 = y^2 = 0$ in $H^2(T)$, and xy generates $H^2(T)$. Then

$$f^*(xy) = (f^*x)(f^*y) = (ax + cy)(bx + dy) = abx^2 + adxy + bcyx + cdy^2 = (ad - bc)xy.$$

So $\deg f = ad - bc = \det F^* = \det F$.

9 Spring 2010

Problem. (S10:01)

Let M_n be the space of $n \times n$ real matrices and let S_n be the subspace of symmetric matrices. Consider the map $F : M_n \to S_n : F(A) = A^t A - I$.

(a) Show that 0 is a regular value of F.

(b) Deduce that $O(n)$, the orthogonal group, is a smooth submanifold of M_n.

(c) Find the dimension of $O(n)$, and determine its tangent space at the identity as a subspace of the tangent space of M_n, which is M_n itself.

Solution.

(a) M_n is a linear space, so $A+sH$ is in M_n for all $s \in \mathbb{R}$ and all $H \in M_n$. S_n can be thought of as $\mathbb{R}^{(n+1)n/2}$, and so $T_0 S_n$ can be identified with S_n. Then

$$\begin{aligned} F(A+sH) &= (A+sH)^t(A+sH) - I \\ &= (A^t + sH^t)(A+sH) - I \\ &= A^tA + sA^tH + sH^tA + s^2H^tH. \end{aligned}$$

Then

$$\frac{1}{s}(F(A+sH) - F(A)) = A^tH + H^tA + sH^tH,$$

and

$$\lim_{s\to 0} \frac{1}{s}(F(A+sH) - F(A)) = A^tH + H^tA.$$

Therefore

$$dF_A(H) = A^tH + H^tA \in S_n.$$

Now suppose $F(A) = 0$. Then $A^tA = I$. We must show that $dF_A : M_n \to S_n$ is surjective. Let $B \in S_n$; we must find $H \in M_n$ such that

$$B = A^tH + H^tA.$$

Take $H = \frac{1}{2}AB$; then

$$A^tH + H^tA = \frac{1}{2}\left[A^tAB + (AB)^tA\right] = \frac{1}{2}\left[B + B^tA^tA\right] = \frac{1}{2}(B + B^t) = B.$$

Thus we have shown dF_A is surjective whenever $F(A) = 0$, so 0 is a regular value of F.

(b) This now follows directly from the regular preimage theorem.

(c) From the regular preimage theorem, we have codim $O(n) = \dim S_n$, so

$$\dim O(n) = n^2 - \frac{(n+1)n}{2} = \frac{n^2 - n}{2}.$$

Note that if $H \in T_I O(n)$, then $c : s \mapsto e^{sH}$ defines for short time s a curve on $O(n)$, with $c(0) = I$ and $c'(0) = H$. Since $e^{sH} \in O(n)$, we have

$$I = (e^{sH})^t(e^{sH}) = e^{sH^t}e^{sH}.$$

Expanding each exponential by its series definition, taking the derivative in s, and letting $s \to 0$, we obtain

$$0 = H^t + H.$$

So $T_I O(n)$ consists of those $H \in M_n$ with $H^t + H = 0$.

Problem. (S10:05) Explain why the following holds: if $\pi : S^N \to M$, $N > 1$ is a (smooth?) covering space with M orientable, then every closed k-form on M, $1 \le k < N$, is exact.

Solution. The covering map induces pullbacks $\pi^* : \Lambda^*(M) \to \Lambda^*(S^N)$, and corresponding maps on de Rham cohomology. The pullback π^* is induced by the pullback of forms

$$\pi^* \omega(X^1, \ldots, X^n) = \omega(\pi_* X^1, \ldots, \pi_* X^n).$$

Since π is a covering map, it is a local diffeomorphism, and therefore π_* is everywhere nonsingular. Suppose now that $\pi^*\omega = 0$. Then since π_* is everywhere nonsingular, and the value of $\omega_p(Y^1, \ldots, Y^n)$ is determined only by $Y^1_p, \ldots, Y^n_p$, we may vary our choice of tangent vectors $X^1_p, \ldots, X^n_p$ to conclude that $\omega = 0$. Therefore $\pi^* : \Lambda^*(M) \to \Lambda^*(S^N)$ is injective.

Now, let $\omega \in \Lambda^k(M)$ be a closed form, $1 \le k < N$. We must exhibit a form $\hat{\omega} \in \Lambda^{k-1}(M)$ with $d\hat{\omega} = \omega$. Since S^n has zero cohomology in these dimensions, $\pi^*[\omega] = [\pi^*\omega] = 0 \in H^k_{dR}(S^N)$. Therefore there exists $\eta \in \Lambda^{k-1}(S^N)$ with $d\eta = \pi^*\omega$.

Define now $\hat{\eta} \in \Lambda^{k-1}(S^N)$ by averaging over the group G of deck transformations:

$$\hat{\eta} = \frac{1}{|G|} \sum_{g \in G} g^* \eta.$$

($|G|$ is finite because S^N is a compact covering space.) This satisfies $d\hat{\eta} = d\eta$. Now we claim $\hat{\eta} = \pi^*\hat{\omega}$ for some $\hat{\omega} \in \Lambda^{k-1}(M)$. To see this, we construct $\hat{\omega}$ pointwise. For $p \in M$, let $\{q_1, \ldots, q_n\} = \pi^{-1}\{p\}$. Pick any $i = 1, \ldots, n$, and set $D_i = (\pi_* : T_{q_i} S^N \to T_p M)$; note that D_i is a vector space isomorphism for each i, because π_* is a local diffeomorphism. Note also that for any $g \in G$, $\pi \circ g = \pi$. So if $g(q_i) = q_j$, then $D_j \circ g_* = D_i$. For $X_1, \ldots, X_{k-1} \in T_p M$, define

$$\hat{\omega}_p(X_1, \ldots, X_{k-1}) = \hat{\eta}_{q_i}((D_i)^{-1} X_1, \ldots, (D_i)^{-1} X_{k-1})$$

for any $i = 1, \ldots, n$. This is well-defined, because of the equivariance of $\hat{\eta}$: if $g \in G$ is a deck transformation with $g(q_i) = q_j$, then since $g^*\hat{\eta} = \hat{\eta}$ we see that

$$\begin{aligned}
\hat{\eta}_{q_i}((D_i)^{-1}X_1, \ldots, (D_i)^{-1}X_{k-1}) &= (g^*\hat{\eta})_{q_i}((D_i)^{-1}X_1, \ldots, (D_i)^{-1}X_{k-1}) \\
&= \hat{\eta}_{q_j}(g_*(D_i)^{-1}X_1, \ldots, g_*(D_i)^{-1}X_{k-1}) \\
&= \hat{\eta}_{q_j}((D_i \circ (g^{-1})_*)^{-1}X_1, \ldots, (D_i \circ (g^{-1})_*)^{-1}X_{k-1}) \\
&= \hat{\eta}_{q_j}((D_j)^{-1}X_1, \ldots, (D_j)^{-1}X_{k-1}).
\end{aligned}$$

Thus the definition is independent of the choice of i. Clearly with this definition, $\pi^*\hat{\omega} = \hat{\eta}$. Finally, to see that $d\hat{\omega} = \omega$, we check:

$$\pi^* d\hat{\omega} = d\pi^*\hat{\omega} = d\hat{\eta} = d\eta = \pi^*\omega.$$

Since π^* is injective on forms, $d\hat{\omega} = \omega$. Thus ω is exact.

10 Fall 2010

Problem. (F10:07)

(a) Let G be a finitely presented group. Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.

(b) Give an example of X in the case $G = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.

(c) How many connected, 2-sheeted covering spaces does the space X from (b) have?

Solution.

(a) Suppose G is presented in the form $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ where $a_1, \ldots, a_n$ are the generators and $r_1, \ldots, r_m$ are the relations. The required space X is constructed as follows: begin with a wedge Y of n circles $C_1, \ldots, C_n$, with each circle C_i corresponding to a generator a_i. The fundamental group of this space is the free group on $a_1, \ldots, a_n$. For each generator a_i, define the map also denoted by a_i to be the homeomorphism and loop $a_i : S^1 \to C_i$. Let a_i^{-1} be the loop corresponding to taking a_i with opposite orientation, and for a given word w in the free group define the corresponding map and loop $w : S^1 \to Y$ by concatenation of loops. To introduce the relations $r_1, \ldots, r_m$, for each j

we attach a disk D^2 to Y along the around $\partial D^2 = S^1$ by the word r_j. Let the resulting space be X. Since D^2 deformation retracts to a point, the loop in X corresponding to r_j contracts to a point, so it is trivial in the fundamental group, as desired.

(b) Take $S^1 \vee S^1$, the wedge of two circles.

(c) This amounts to counting the number of index two subgroups of $\mathbb{Z} * \mathbb{Z}$. Note that such a subgroup N is normal, and for each such subgroup there is a homomorphism

$$\phi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$

with kernel N. Let a and b be the generators of $\mathbb{Z} * \mathbb{Z}$. Regarding $\mathbb{Z}/2\mathbb{Z}$ as the additive group of integers modulo 2, there are four possible homomorphisms $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ we could define:

i. $a, b \mapsto 0$. The kernel is $\ker \phi = \mathbb{Z} * \mathbb{Z}$.

ii. $a \mapsto 1$, $b \mapsto 0$. The kernel is $\langle a^2, b, aba^{-1} \rangle$.

iii. $a \mapsto 0$, $b \mapsto 1$. The kernel is $\langle a, b^2, bab^{-1} \rangle$.

iv. $a, b \mapsto 1$. The kernel is $\langle a^2, b^2, ab \rangle$.

Of these, ii, iii, and iv are index 2, while i is obviously not. ii and iii produce isomorphic covers. (See Hatcher p.58 for some pictures.) Thus there are two connected 2-sheeted covering spaces up to isomorphism.

Problem. (F10:08) Let G be a connected topological group. Show that $\pi_1(G)$ is a commutative group.

Solution. (Remark: This is known as the Eckman-Hilton argument.)

Let us take $e \in G$ to be the basepoint of G. For any two loops $\gamma_1, \gamma_2 : S^1 \to G$ based at e, we define the loop $\gamma_1 * \gamma_2$ by

$$\gamma_1 * \gamma_2(t) = \gamma_1(t) \cdot \gamma_2(t),$$

where $\cdot$ denotes the group multiplication. We claim that $*$ defines a group structure on $\pi_1(G)$ that agrees with the standard group structure. This is because up to equivalence of loops, we are free to reparametrize our loops γ_1 and γ_2. Parametrize γ_1 so that $\gamma_1(t) = e$ for $t \in [\frac{1}{2}, 1]$, and γ_2 so that $\gamma_2(t) = e$ for $t \in [0, \frac{1}{2}]$. Then $\gamma_1 * \gamma_2$ is the same as the concatenation of γ_1 and γ_2, that is, going around γ_1

first and γ_2 second. Therefore taking equivalence classes of loops we find that $[\gamma_1 * \gamma_2] = [\gamma_1][\gamma_2]$. Clearly the constant loop acts as the identity under $*$ as well, and the backwards oriented loop acts as inversion for $*$ by the same argument that $*$ is equivalent to concatenation.

To see that $\pi_1(G)$ is a commutative group, we note that we can just as well reparametrize our loops in another way: take $\gamma_1(t) = e$ for $t \in [0, \frac{1}{2}]$, and $\gamma_2(t) = e$ for $t \in [\frac{1}{2}, 1]$. Then $\gamma_1 * \gamma_2 = \gamma_2\gamma_1$, that is, going around γ_2 first followed by γ_1. Thus we find that $[\gamma_2][\gamma_1] = [\gamma_1][\gamma_2]$.

Problem. (F10:09) Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

Solution. If $F : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism, then it is also a homeomorphism $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{F(0)\}$. Each of these spaces is homotopy equivalent to S^{m-1} and S^{n-1} respectively. Therefore in homology we find that $H_k(S^{m-1}) \cong H_k(S^{n-1})$ for all k. This can only be true if $m = n$.

Problem. (F10:10) Let N_g be the nonorientable surface of genus g, that is, the connected sum of g copies of $\mathbb{RP}^2$. Find the fundamental group and homology groups of N_g.

Solution.

(a) Fundamental group: N_g can be given the following cell structure:

  i. One 0-cell v.
  ii. g 1-cells $a_1, \dots, a_g$ each attached to v along endpoints.
  iii. One 2-cell f attached to the edges along the word $a_1^2 \cdots a_g^2$.

  Then for the fundamental group we have the presentation

$$\pi_1(N_g) = \langle a_1, \dots, a_g \mid a_1^2 \cdots a_g^2 \rangle.$$

(b) We use cellular homology. The cellular chain complex is

$$0 \longrightarrow \mathbb{Z}\langle f \rangle \cong \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}\langle a_1, \dots, a_g \rangle \cong \mathbb{Z}^g \xrightarrow{\partial_1} \mathbb{Z}\langle v \rangle \cong \mathbb{Z} \longrightarrow 0$$

  $H_0(N_g) \cong \mathbb{Z}$ because N_g is connected. The cellular boundary maps are given on the bases by

$$\partial_2(f) = 2a_1 + \cdots + 2a_g,$$
$$\partial_1(a_i) = v - v = 0.$$

So we see that $\ker \partial_2 = 0$, hence $H_2(N_g) = 0$. $\text{im } \partial_2 = 2\mathbb{Z}\langle a_1, \ldots, a_g \rangle$, while $\ker \partial_1 = \mathbb{Z}\langle a_1, \ldots, a_g \rangle$. Therefore

$$H_1(N_g) = \frac{\ker \partial_1}{\text{im } \partial_2} = \frac{\mathbb{Z}\langle a_1, \ldots, a_g \rangle}{2\mathbb{Z}\langle a_1, \ldots, a_g \rangle}.$$

To determine the structure of this quotient, we endow $\mathbb{Z}\langle a_1, \ldots, a_g \rangle$ with an alternative basis, consisting of

$$a_1, \ldots, a_{g-1}, a_1 + \cdots + a_g.$$

Then the quotient collapses the free submodule generated by $a_1 + \cdots + a_g$ and leaves the rest alone, so

$$\frac{\mathbb{Z}\langle a_1, \ldots, a_g \rangle}{2\mathbb{Z}\langle a_1, \ldots, a_g \rangle} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

(Alternatively, apply the Hurewicz homomorphism and abelianize $\pi_1(N_g)$.)

11 Spring 2011

Problem. (S11:02)

(a) Demonstrate the formula
$$\mathcal{L}_X = di_X + i_X d,$$
where $\mathcal{L}$ is the Lie derivative and i is the interior product.

(b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a flow that (locally) preserves volume if and only if the divergence of X is everywhere 0.

Solution.

(a) This is one of Cartan's formulas. No tricks here, just careful writing of definitions; see Morita or Lee.

(b) Consider the standard volume form $\omega = dx \wedge dy \wedge dz$ on $\mathbb{R}^3$. We are being asked to show that $\mathcal{L}_X \omega = 0$ if and only if $\div X \equiv 0$. By Cartan's formula we compute
$$\mathcal{L}_X \omega = di_X \omega + i_X d\omega = di_X \omega.$$

Write $X = f\partial_x + g\partial_y + h\partial_z$. By linearity, and the definitions for $dx, dy, dz, \partial_x, \partial_y, \partial_z$, we have

$$\begin{aligned} i_X\omega(Y,Z) &= f\omega(\partial_x, Y, Z) + g\omega(\partial_y, Y, Z) + h\omega(\partial_z, Y, Z) \\ &= f(dy(Y)dz(Z) - dz(Y)dy(Z)) + g(dz(Y)dx(Z) - dx(Y)dz(Z)) \\ &\quad + h(dx(Y)dy(Z) - dy(Y)dz(Z)) \\ &= f(dy \wedge dz)(Y,Z) + g(dz \wedge dx)(Y,Z) + h(dx \wedge dy)(Y,Z). \end{aligned}$$

Thus $i_X\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$, and so

$$di_X\omega = (f_x + g_y + h_z) dx \wedge dy \wedge dz.$$

We thus have

$$\mathcal{L}_X(dx \wedge dy \wedge dz) = \operatorname{div} X \cdot dx \wedge dy \wedge dz.$$

From here it is clear that $\mathcal{L}_X(dx \wedge dy \wedge dz) = 0$ if and only if $\operatorname{div} X \equiv 0$.

Problem. (S11:09)

(a) State the Lefschetz fixed point theorem.

(b) Show that the Lefschetz number of any map from $\mathbb{CP}^{2n}$ to itself is nonzero, and hence every map from $\mathbb{CP}^{2n}$ to itself has a fixed point.

Solution.

(a) We state the homological version, which can be found in Hatcher.

Definition. Let A be a finitely generated abelian group with torsion subgroup Tor. Let $f : A \to A$ be a homomorphism. Then the *trace* of f, denoted $\operatorname{tr}(f)$, is defined to be the trace of $f : A/\operatorname{Tor} \to A/\operatorname{Tor}$.

Lefschetz fixed point theorem. Let X be a topological space whose homology groups are finitely generated and vanish for dimensions $> N$. For a continuous map $F : X \to X$ define the *Lefschetz number* $\tau(F)$ by

$$\tau(F) = \sum_{n=0}^{N} (-1)^n \operatorname{tr}(F_* : H_n(X) \to H_n(X))$$

where $F_* : H_n(X) \to H_n(X)$ is the induced map on homology. If $\tau(F) \neq 0$, then F has a fixed point.

(b) Let $F : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ be a continuous map. The homology of $\mathbb{CP}^{2n}$ is $\mathbb{Z}$ in even dimensions, 0 in odd dimensions. Since the homology groups are free and finitely generated, by universal coefficients the integer cohomology groups are dual to the integer homology groups, and $F^* : H^k(\mathbb{CP}^{2n}) \to H^k(\mathbb{CP}^{2n})$ is the transpose of $F_* : H_k(\mathbb{CP}^{2n}) \to H_k(\mathbb{CP}^{2n})$. In particular, these maps have the same trace. Therefore

$$\begin{aligned}\tau(F) &= \sum_{k=0}^{4n} (-1)^k \mathrm{tr}(F_* : H_k(\mathbb{CP}^{2n}) \to H_k(\mathbb{CP}^{2n})) \\ &= \sum_{k=0}^{4n} (-1)^k \mathrm{tr}(F^* : H^k(\mathbb{CP}^{2n}) \to H^k(\mathbb{CP}^{2n})) \\ &= \sum_{j=0}^{2n} \mathrm{tr}(F^* : H^{2j}(\mathbb{CP}^{2n}) \to H^{2j}(\mathbb{CP}^{2n})).\end{aligned}$$

Now, we note that the cohomology ring of $\mathbb{CP}^{2n}$ is generated by a generator of $H^2(\mathbb{CP}^{2n})$. Let a be such a generator, and let M be an integer with $F^*a = Ma$. Then

$$F^*(a^j) = (F^*a)^j = M^j a^j,$$

and since each $2j$-th degree cohomology group is $\mathbb{Z}$, this gives

$$\mathrm{tr}(F^* : H^{2j}(\mathbb{CP}^{2n}) \to H^{2j}(\mathbb{CP}^{2n})) = M^j.$$

Therefore

$$\tau(F) = \sum_{j=0}^{2n} M^j.$$

In particular this number is never 0, for any choice of integer M: the only choice that could possibly work is $M = -1$, but because there are an odd number of terms this cannot add to 0.

12 Fall 2011

Problem. (F11:01) Let M be an (abstract) compact smooth manifold. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be smoothly embedded in the Euclidean space $\mathbb{R}^n$.

Solution. The following is from Lee. Let $\dim M = N$. Since M is compact, M admits a finite cover by regular coordinate balls or half-balls $B_1, \ldots, B_k$: that is, for each i there is a coodinate domain $B_i' \supset \overline{B_i}$ such that the coordinate map $\varphi_i : B_i' \to \mathbb{R}^N$ restricts to a diffeomorphism of $\overline{B_i}$ to a compact subset of $\mathbb{R}^N$. For each i, let $\rho_i : M \to \mathbb{R}$ be a smooth bump function that is 1 on $\overline{B_i}$ and supported in B_i'. Define $F : M \to \mathbb{R}^{Nk+k}$ by

$$F(p) = (\rho_1(p)\varphi_1(p), \ldots, \rho_k(p)\varphi_k(p), \rho_1(p), \ldots, \rho_k(p))$$

where $\rho_i\varphi_i$ is extended by 0 outside of supp ρ. F is clearly smooth; we claim F is an injective immersion. Since M is compact, an injective immersion is automatically an embedding, so this suffices to prove the claim.

To see that F is injective, suppose $F(p) = F(q)$. p lies in some B_i, so $\rho_i(p) = 1$. Then $\rho_i(q) = 1$ as well. Then

$$\varphi_i(p) = \rho_i(p)\varphi_i(p) = \rho_i(q)\varphi_i(q) = \varphi_i(q).$$

Since we took φ_i to be diffeomorphisms on $\overline{B_i}$, we see that $p = q$.

To see that F is an immersion, let $p \in M$. Then $p \in B_i$ for some i. Since $\rho_i \equiv 1$ on B_i, $d(\rho_i\varphi_i)_p = d(\varphi_i)_p$, which is injective. From this it follows that dF_p is also injective.

Problem. (F11:03) Let M be a compact, simply connected smooth manifold of dimension n. Prove that there is no smooth immersion $f : M \to T^n$, where T^n is the n-torus.

Solution. Suppose f is a smooth immersion $f : M \to T^n$. Since $\dim M = n$, f must be a submersion as well, hence a local diffeomorphism. Since M is compact, $f(M)$ is also compact, and in particular closed. Since f is a local diffeomorphism, $f(M)$ is also open. Therefore $f(M) = T^n$, and f is surjective. Since M is compact, the preimages of compact sets are compact, so f is a proper map. A surjective proper local diffeomorphism from a compact manifold is a covering map (stack of records theorem). But M is simply connected, while the universal cover of T^n is $\mathbb{R}^n$. Since the universal cover is unique up to isomorphism, there must be a covering space isomorphism $\mathbb{R}^n \to M$. However, a covering space isomorphism is in particular a homeomorphism, and M is compact while $\mathbb{R}^n$ is not, so in fact there does not even exist a homeomorphism $\mathbb{R}^n \to M$, contradiction.

Problem. (F11:04) Give a topological proof of the Fundamental Theorem of Algebra.

Solution. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Suppose, aiming for contradiction, that p has no zeros. In particular we may take $a_0 \neq 0$. Let C_R denote the closed curve $p(Re^{i\theta})$, $\theta \in [0, 2\pi]$, and let w_R denote the winding number of C_R around 0. For $|z|$ sufficiently large, we note that

$$\left| \sum_{i=0}^{n-1} a_i z^i \right| \ll |z|^n,$$

and therefore if R is sufficiently large than the winding number of C_R is equal to the winding number of the curve $(Re^{i\theta})^n = R^n e^{in\theta}$ around 0; thus $w_R = n$ for R large. On the other hand, when R is sufficiently small, then the dominant term in the polynomial is a_0, and so for R small w_R is equal to the winding number of the constant curve at a_0; hence $w_R = 0$ for R small. But winding number is invariant under admissible homotopies of the curve, and since p has no zeros by hypothesis the radial homotopy provides an obvious admissible homotopy between all curves C_R. But then we conclude that $w_R = n = 0$, contradiction for $n \geq 1$.

Problem. (F11:05) Let $f : M \to N$ be a smooth map between two manifolds M and N. Let α be a p-form on N. Show that $d(f^*\alpha) = f^*(d\alpha)$.

Solution. It suffices to prove the claim in local coordinates. Let x be local coordinates on N. Assume without loss of generality that

$$\alpha = g dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Then since $d^2 = 0$,

$$d\alpha = dg \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Then

$$\begin{aligned} f^*(d\alpha) &= (f^*dg) \wedge (f^*dx^{i_1}) \wedge \cdots \wedge (f^*dx^{i_p}) \\ &= d(g \circ f) \wedge (d(x^{i_1} \circ f)) \wedge \cdots \wedge (d(x^{i_p} \circ f)) \end{aligned}$$

and using $d^2 = 0$ again,

$$\begin{aligned} d(f^*\alpha) &= d((g \circ f) d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_p} \circ f)) \\ &= d(g \circ f) \wedge d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_p} \circ f)). \end{aligned}$$

Problem. (F11:08)

(a) Let M be a Möbius band. Using homology, show that there is no retraction from M to ∂M.

(b) Let K be a Klein bottle. Show that there exist homotopically nontrivial simple closed curves γ_1 and γ_2 on K such that K retracts to γ_1, but does not retract to γ_2.

Solution.

(a) See F13:02 for a more general version not using homology. Suppose $r : M \to \partial M$ is a retract, and let ι be the inclusion map $\iota : \partial M \hookrightarrow M$. Then r and ι induce maps on singular homology, and $r_*\iota_* = \mathrm{id}_{H_*(\partial M)}$. But notice that if $a \in H_1(\partial M)$ is the homology class of ∂M, then because ∂M goes around the central circle of the Möbius band twice it follows that $\iota_*(a) = 2b$, where b is the class of the middle circle of $H_1(M)$. a generates $H_1(\partial M) \cong \mathbb{Z}$ and b generates $H_1(M) \cong \mathbb{Z}$. We thus find that $a = r_*\iota_*(a) = 2r_*(b)$. But this cannot be, because r_* is surjective, contradiction.

(b) ?

Problem. (F11:07) Consider the form

$$\omega = (x^2 + x + y)dy \wedge dz$$

on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ the inclusion.

(a) Calculate $\int_{S^2} \omega$.

(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

Solution.

(a) We compute

$$d\omega = (2x + 1)dx \wedge dy \wedge dz.$$

Let B^3 denote the closed unit ball in $\mathbb{R}^3$. By Stokes' theorem

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega = \int_{B^3} (2x+1)dx \wedge dy \wedge dz.$$

Noting that x is odd, by symmetry the integral of $2xdx \wedge dy \wedge dz$ over B^3 is zero. Therefore we obtain

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega = \int_{B^3} dx \wedge dy \wedge dz = \mathrm{Vol}(B^3).$$

(b) Such a form cannot exist. For suppose α is such a form. By the Poincaré lemma, every closed form on $\mathbb{R}^3$ is also exact, so $\alpha = d\eta$ for some 1-form η on $\mathbb{R}^3$. Then

$$\int_{S^2} i^*\alpha = \int_{S^2} di^*\eta = \int_{\partial S^2} i^*\eta = \int_{\varnothing} i^*\eta = 0.$$

On the other hand we know from (a) that the integral of $i^*\omega$ over S^2 is not zero, contradiction.

Problem. (F11:09) Let X be the topological space obtained from a pentagon by identifying its edges in a cycle (see actual exam for picture). Calculate the homology and cohomology groups of X with integer coefficients.

Solution. The CW complex structure on X has one 0-cell v, one 1-cell e attached to v along both ends, and one 2-cell f that is attached to the 1-skeleton along the word e^5. The cellular chain complex associated to this is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

and we end up with the homology groups

$$H_0(X) = \mathbb{Z},\ H_1(X) = \mathbb{Z}/5\mathbb{Z},\ H_2(X) = 0.$$

Dualizing the cellular chain complex gives us the cochain complex

$$0 \longleftarrow \mathbb{Z} \xleftarrow{5} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

and we thus obtain cohomology groups

$$H^0(X) = \mathbb{Z},\ H^1(X) = 0,\ H^2(X) = \mathbb{Z}/5\mathbb{Z}.$$

Problem. (F11:10) Let X, Y be topological space and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $Y \coprod (X \times [0,1])$ by identifying $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Solution. See S15:09.

13 Spring 2012

Problem. (S12:09)
Suppose a finite group Γ acts smoothly on a compact manifold M and that the faction is free, i.e. $\gamma(x) = x$ for some $x \in M$ if and only if γ is the identity element of Γ.

(a) Show that M/Γ is a manifold (i.e. can be made a manifold in a natural way).

(b) Show that $M \to M/\Gamma$ is a covering space.

(c) If the k-th de Rham cohomology of M is 0, some particular $k > 0$, is the k-th de Rham cohomology of M/Γ necessarily 0? Prove your answer.

Solution.
(a) and (b) are standard results about quotient manifolds, but the proof of (a) is quite involved; see Lee Chapter 21. As for (c), suppose $H^k_{dR}(M) = 0$. We can then follow the argument of S10:05 to show that every closed form in $\Omega^k(M/\Gamma)$ is exact. Therefore it follows that $H^k_{dR}(M/\Gamma) = 0$ as well.

14 Fall 2012

Problem. (F12:02) For $n \geq 1$, construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space $\mathbb{RP}^{2n-1}$.

Solution. Odd-dimensional projective spaces are compact, orientable, and have Euler characteristic zero. Therefore it suffices to show that all compact connected orientable smooth manifolds with zero Euler characteristic admit a nonvanishing smooth vector field. Let M be such a manifold of dimension N, and let X be any smooth vector field on M with isolated zeros. By compactness there can only be finitely many, and by the Poincaré-Hopf theorem we know that

$$\sum_{p:X_p=0} \operatorname{ind}(X,p) = 0.$$

Since M is connected, there is a coordinate chart U containing all the zeros of X, and we may choose U such that $\partial U \simeq S^{N-1}$. Then since X does not vanish on ∂U, there is an induced map $f : S^{N-1} \to S^{N-1} : p \mapsto X_p/|X_p|$ (where we give M a Riemannian structure, so that $|X_p|$ is well-defined). The degree of f is equal to the sum of the indices of the zeros of X contained in U; since every zero of

X is contained in U, f is a degree zero map. Therefore f extends to a map $f : U \to S^{N-1}$: that is, there is a vector field Y on $\overline{U}$ such that $|Y_p| = 1$ on $\overline{U}$, and $Y_p = X_p/|X_p|$ on ∂U. Then we may extend Y_p to M by setting $Y_p = X_p/|X_p|$ for all $p \notin U$, since X_p is nonvanishing outside of U. Then Y_p is the desired nonvanishing vector field on M.

Problem. (F12:03)
Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

Solution.
See F14:02 for the assertion that $\mathbb{R}^n \setminus M$ is connected. Now suppose $f : S^1 \to \mathbb{R}^n \setminus M$ is a loop. By Whitney approximation we may suppose that f is smooth up to homotopy. Since $\mathbb{R}^n$ is simply connected, there is a homotopy that contracts f to a point; again, up to homotopy this homotopy is smooth. Finally, we may perturb f via a small homotopy so that f is actually a smooth embedding. Then the contraction of f to a point sweeps out a smooth surface S with boundary. Since transversality is generic, we may smoothly perturb this homotopy so that S intersects M transversely. Then by the dimension-counting argument of F14:02 , S does not intersect M. Therefore we have produced a homotopy of f to the constant loop that stays within $\mathbb{R}^n \setminus M$; this shows that $\mathbb{R}^n \setminus M$ is simply connected.

Problem. (F12:04)

1. Show that for any $n \geq 1$ and $k \in \mathbb{Z}$, there exists a continuous map $f : S^n \to S^n$ of degree k.

2. Let X be a compact, oriented n-manifold. Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : X \to S^n$ of degree k.

Solution.
We may as well prove (b) only. First take $k \geq 0$. Let $x_1, \ldots, x_k$ be distinct points in X, and let $B_1, \ldots, B_k \subset X$ be small disjoint coordinate neighborhoods diffeomorphic to $\mathbb{D}^n$ around $x_1, \ldots, x_k$ respectively. Let N and S denote the north and south poles of S^n respectively. Then we construct f as follows. Define $f(x) = S$ for all $x \notin \bigcup B_i$. As for $x \in B_i$, let $h : \mathbb{D}^n \to S^n$ be a map sending 0 to N and mapping diffeomorphically onto $S^n \setminus \{S\}$, preserving orientations. (This is easily accomplished by, say, stereographic projection). Then arranging the coordinate neighborhoods so that x_i corresponds to 0 in each coordinate chart, we obtain a

map $B_i \to S^n$ sending x_i to N and mapping the rest of B_i diffeomorphically onto $S^n \setminus \{N,S\}$. Let f be this map on $\bigcup B_i$. Then f has degree k. For

$$\deg f = \sum_{p \in f^{-1}(N)} \operatorname{sgn} Jf(p).$$

$f^{-1}(N)$ consists of $\{x_1,\dots,x_k\}$, and at each x_i f is an orientation-preserving diffeomorphism, so $\deg f = k$. To extend this to k negative, carry out the above construction for $-k$, but map $\mathbb{D}^n$ to S^n in an orientation-reversing manner; this can be done by precomposing a stereographic projection with a reflection. Then f is orientation-reversing at each x_i, so $\deg f = -(-k) = k$.

Problem. (F12:05)
Assume that $\Delta = \{X_1,\dots,X_k\}$ is a k-dimensional distribution spanned by vector fields on an open set $\Omega \subset M^n$ in an n-dimensional manifold. For each open set $V \subset \Omega$ define

$$\mathcal{Z}_V = \{u \in C^\infty(V) : X_1 u = \cdots = X_k u = 0\}.$$

Show that the following are equivalent:

(a) Δ is integrable.

(b) For each $x \in \Omega$ there exists an open neighborhood $x \in V \subset \Omega$ and $n-k$ functions $u_1,\dots,u_{n-k} \in \mathcal{Z}_V$ such that the differentials $du_1,\dots,du_{n-k}$ are linearly independent at each point of V.

Solution.
Suppose Δ is integrable. Then through $p \in \Omega$, Δ has an integral submanifold $N = M \cap V$, whose tangent bundle is spanned by $\{X_i\,|_V\}$. Since N has codimension $n-k$, N is locally cut out as the locus of $n-k$ functions $u_1,\dots,u_{n-k}$ with linearly independent differentials (implicit function theorem). Since u_i are constant on N, it follows that $X_i u_j = 0$ for all i,j; this produces the desired functions.

Now suppose (b) holds. We claim that the ideal $\mathcal{I}(\Delta)$ of forms vanishing on Δ is a differential ideal; then the claim follows by Frobenius. Note that it suffices to show this locally, and for this to hold in $V \ni x$ it suffices to show that the ideal generated by $\{du_1,\dots,du_{n-k}\}$ (where u_i are as given by the hypothesis (b)) is equal to $\mathcal{I}(\Delta)$: this is because if ω is a form generated by $du_1,\dots,du_{n-k}$, i.e.

$$\omega = \sum_{i=1}^{n-k} \theta_i \wedge du_i,$$

then

$$d\omega = \sum_{i=1}^{n-k} d\theta_i \wedge du_i + (-1)^{\deg \theta_i} \theta_i \wedge ddu_i = \sum_{i=1}^{n-k} d\theta_i \wedge du_i$$

is also generated by $du_1, \ldots, du_{n-k}$. Thus this ideal is a differential ideal.

To see that the ideal $\mathcal{J}$ generated by $du_1, \ldots, du_{n-k}$ coincides with $\mathcal{I}(\Delta)$, first suppose $\omega \in \mathcal{J}$, and write $\omega = \sum_{i=1}^{n-k} \theta_i \wedge du_i$. Then ω vanishes on Δ, as a consequence of the fact that $du_i(X_j) = X_j(u_i) = 0$. Thus $\mathcal{J} \subset \mathcal{I}(\Delta)$. Equality now follows from dimension-counting, because $\mathcal{J}$ and $\mathcal{I}(\Delta)$ are both generated by $n-k$ linearly independent 1-forms.

Problem. (F12:07) Let $n \geq 0$ be an integer. Let M be a compact (closed?), orientable smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Solution.
Same problem as S15:10.

Problem. (F12:08)
Show that there is no compact three-manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

Solution.
Suppose for contradiction that such M exists, and assume for now that M is orientable. Then we may construct an orientable closed 3-manifold N by taking taking the disjoint union $M \coprod M$ and gluing their boundaries along the identity map $\mathrm{id} : \mathbb{RP}^2 \to \mathbb{RP}^2$. Then there exist open subsets $M_1, M_2 \subset N$ that each retract to a copy of M, and whose intersection $M_1 \cap M_2$ retracts to $\mathbb{RP}^2$ (basically take a small neighborhood of each copy of M in N). Then for this open cover of N we have the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^n_{dR}(N) \longrightarrow H^n_{dR}(M) \oplus H^n_{dR}(M) \longrightarrow H^n_{dR}(\mathbb{RP}^2) \longrightarrow \cdots$$

which implies $\chi(N) - 2\chi(M) + \chi(\mathbb{RP}^2) = 0$. By Poincaré duality, since N is a closed connected orientable 3-manifold, it follows that $\chi(N) = 0$, and thus we find that $\chi(\mathbb{RP}^2)$ is even. But $\chi(\mathbb{RP}^2) = 1$, contradiction. If N happens to be nonorientable (which could arise when M is nonorientable), then we consider the orientation double cover $\tilde{N} \to N$. Since this is a 2-fold cover, we have $\chi(\tilde{N}) = 2\chi(N)$. Since $\tilde{N}$ is a closed connected orientable 3-manifold, $\chi(\tilde{N}) = 0$, so $\chi(N) = 0$ and the rest of the proof follows through.

Problem. (F12:09) Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \dots, x_n) : x_j = 0 \text{ for all } j \neq i\}.$$

Calculate the homology groups of the complement $\mathbb{R}^n \setminus (L_1 \cup \cdots \cup L_n)$.

Solution.
This is basically the same problem as S16:08 , and the same observation makes this problem trivial.

Problem. (F12:10)

(a) Let X be a finite CW complex. Explain how the homology groups of $X \times S^1$ are related to those of X.

(b) For each integer $n \geq 0$, give an example of a compact smooth manifold of dimension $2n+1$ such that $H_i(X) = \mathbb{Z}$ for $i = 0, \dots, 2n+1$.

Solution.

(a) We use cellular homology. Let $S_n = X^n \setminus X^{n-1}$ be the collection of n-cells of X. Give S^1 the standard CW structure with one 0-cell v and one 1-cell e. Then the product $X \times S^1$ has the product CW complex structure, whose cells are products of cells in X and S^1. For any given n, the n-cells of $X \times S^1$ consist of the following:

   i. Products of n-cells in X and v, and:
   ii. Products of $n-1$-cells in X and the e.

   In general, given two CW complexes Y and Z, y an m-cell of Y, and z an n-cell of Z, the boundary homomorphism on the product works like

$$\partial_{m+n}(y,z) = (\partial_m y, z) + (-1)^m (y, \partial_n z).$$

   For S^1, the boundary maps are

$$\partial_0(v) = 0,\ \partial_1(e) = v - v = 0.$$

   Therefore for n-cells in $X \times S^1$,

$$\partial_n(e^n, v) = (\partial_n e^n, v) + (-1)^n (e^n, 0) = (\partial_n e^n, v),$$

$$\partial_n(e^{n-1},e) = (\partial_{n-1}e^{n-1},e) + (-1)^{n-1}(e^{n-1},0) = (\partial_{n-1}e^{n-1},e).$$

Since the (e^n_α, v) and (e^{n-1}_α, e) generate the n-cells, the kernel of ∂_n is generated by those n-cells that are 0 above. Therefore $\ker \partial_n$ is generated by pairs (e^n_α, v) and (e^{n-1}_α, e) with $\partial_n e^n_\alpha = 0$ and $\partial_{n-1} e^{n-1}_\alpha = 0$. Similarly im ∂_n is generated by those pairs (a,v) and (b,e) with $a \in \text{im } \partial_{n+1}$ and $b \in \partial_n$. Therefore

$$H_n(X\times S^1) = \frac{Z_n(X\times S^1)}{B_n(X\times S^1)} \cong \frac{Z_n(X)\oplus Z_{n-1}(X)}{B_n(X)\oplus B_{n-1}(X)} \cong H_n(X)\oplus H_{n-1}(X).$$

(b) $\mathbb{CP}^n \times S^1$ will do.

15 Spring 2013

Problem. (S13:01) Let $\text{Mat}_{m\times n}(\mathbb{R})$ be the space of $m\times n$ matrices with real valued coefficients.

(a) Show that the subset $S \subset \text{Mat}_{m\times n}(\mathbb{R})$ of rank 1 matrices form a submanifold of dimension $m+n-1$.

(b) Show that the subset $T \subset \text{Mat}_{m\times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m+n-k)$.

Solution. Same question as S15:01.

Problem. (S13:02) Let M be a smooth manifold and $\omega \subset \Omega^1(M)$ a smooth 1-form.

(a) Define the line integral

$$\int_c \omega$$

along piecewise smooth curves $c : [0,1] \to M$.

(b) Show that $\omega = df$ for a smooth function $f : M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0,1] \to M$.

Solution.

(a) First, suppose c is smooth, injective, and regular, i.e. $\dot{c}(t) \neq 0$ for all t. Then $c([0,1])$ is an embedded 1-manifold with boundary. For such c, $\int_c \omega$ is defined as follows: for each $t \in [0,1]$, let U_t be a coordinate neighborhood of $c(t)$. By compactness there is a finite sequence $0 = t_0 < \cdots < t_N = 1$ so that $U_{t_0}, \ldots, U_{t_N}$ cover $c([0,1])$. Let $1 = f_1 + \cdots + f_N$ be a smooth partition of unity subordinate to the cover $\{U_{t_i}\}$ of $c([0,1])$. Write $V_i = U_{t_i} \cap c([0,1])$. We then define

$$\int_c \omega = \sum_{i=1}^{N} f_i \int_{c^{-1}(V_i)} c^* \omega.$$

For a general smooth c, reparametrize so that c is regular, and break c into pieces on which c is injective. This misses finitely many points, but such points contribute a set of measure zero, so they do not contribute to the integral. Define $\int_c \omega$ as the sum of the integral of ω over such pieces. Finally, when c is merely piecewise smooth, define $\int_c \omega$ as the sum of the integral of ω over the smooth pieces.

(b) If $\omega = df$, and then by Stokes' theorem it follows that

$$\int_c \omega = \int_c df = \int_{\partial c} f = \int_{\varnothing} f = 0.$$

Conversely, if $\int_c \omega = 0$ for all closed curves c, then the line integral $\int_c \omega$ for arbitrary piecewise smooth curves c depends only on the endpoints of c. We may therefore define a function f as follows: fix a base point p, and set

$$f(q) = \int_{c(p,q)} \omega,$$

where $c(p,q) : [0,1] \to M$ is any piecewise smooth path with $c(0) = p$ and $c(1) = q$. (This, of course, requires that M is path-connected, but this should be implicit in the problem statement anyway.) Then $df = \omega$. For if we choose a coordinate neighborhood (U,x) of q, then by path independence we can take c within this coordinate neighborhood to be a polygonal path that moves parallel to the coordinate directions. In particular, by taking a smaller coordinate neighborhood if necessary, we can take c to be a coordinate curve $c(t) = x_1(t)$, $(x_1(0), \ldots, x_n(0)) = q$, and then by the fundamental theorem of calculus

$$df_q = \left.\frac{d}{dt}\right|_{t=0} \int_{c(t)} \omega = (x_1)'(0)\omega(q) = \omega(q).$$

Problem. (S13:03) Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

(a) Define what it means for S_1, S_2 to be transversal.

(b) Show that if $S_1, S_2 \subset M$ are transversal then $S_1 \cap S_2$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

Solution.

(a) S_1 and S_2 are transversal if

$$T_pS_1 + T_pS_2 = T_pM$$

for all $p \in S_1 \cap S_2$.

(b) Write $\dim M = m$, $\dim S_1 = s_1$, $\dim S_2 = s_2$. Let $F : S_1 \to M$ be the embedding of S_1 into M. Let $p = F(q) \in S_1 \cap S_2$. By the implicit function theorem, near p we may write S_2 as the zero locus of $m - s_2$ independent functions $g_1, \dots, g_{m-s_2} : U \to \mathbb{R}$. Then in a neighborhood W of q, we may write $W \cap F^{-1}(S_2) \subset S_1 \cap S_2$ as the zero locus of $g_i \circ F$, $i = 1, \dots, m - s_2$. Let $g = (g_1, \dots, g_{m-s_2}) : S_2 \to \mathbb{R}^{m-s_2}$. Then g is a submersion, and $g \circ F : W \to \mathbb{R}^{m-s_2}$ is another submersion (since F is an embedding). Therefore 0 is a regular value of $g \circ F$, and $(g \circ F)^{-1}(0) = W$ is a submanifold of S_1 of codimension $m - s_2$, i.e. of dimension $s_1 + s_2 - m$. Since a manifold structure is determined locally, we conclude that $S_1 \cap S_2$ is a submanifold of S_1 of dimension $s_1 + s_2 - m$.

Problem. (S13:04) Let $S \subset M$ be given as $F^{-1}(c)$ where $F = (F^1, \dots, F^k) : M \to \mathbb{R}^k$ is smooth and $c \in \mathbb{R}^k$ is a regular value for F. If $f : M \to \mathbb{R}$ is smooth, show that its restriction $f|_C$ to a submanifold $C \subset M$ has a critical point at $p \in C$ if and only if there exist constant $\lambda_1, \dots, \lambda_k$ such that

$$df_p = \sum \lambda_i dF_p^i$$

where $dg_p : T_pM \to \mathbb{R}$ denotes the differential of a smooth function g.

Solution.
(Remark: this is the Lagrange multiplier theorem.)

The wording of this question makes no sense, because presumably something about S and the regularity of c needs to be used, but the assertion to be proved makes no reference to either. We will therefore assume the examiners intended to write $C = S$.

Write $m = \dim M$, so that $\dim S = n - k$. Let $p \in S$, and suppose p is critical for $f|_S$; that is, $d(f|_S)_p : T_pS \to \mathbb{R}$ is the zero map. In particular, df is a 1-form that vanishes on T_pS. Consider the vector space

$$V = \{\omega \in T_p^*M : T_pS \subset \ker \omega\}.$$

We claim that V has dimension k, and is in particular spanned by dF_p^i, $i = 1, \ldots, k$. That $dF_p^i \in V$ is obvious, because F is constant on S, and therefore dF_p^i acts trivially on T_pS. They are also linearly independent because $S = F^{-1}(c)$ and c is a regular value of F. To see that $\dim V = k$, we note that TS is a k-dimensional distribution on M, and therefore the ideal of forms vanishing on TS is locally generated by $n - (n-k) = k$ independent 1-forms. Thus it follows that $\dim V = k$, and the required assertion follows.

For the converse, suppose $p \in S$ and $df_p = \sum \lambda_i dF_p^i$. Then as previously noted, dF_p^i is zero on T_pS. So clearly df_p is also zero on T_pS, which is to say p is critical for $f|_S$.

Problem. (S13:05) Let M be a smooth orientable compact manifold with boundary ∂M. Show that there is no smooth retract $r : M \to \partial M$.

Solution. See also F13:02 for the case of general manifolds. Let $n = \dim M$. Since ∂M is a closed orientable $(n-1)$-manifold, ∂M has an orientation form ω. Via the retraction r, we define an $(n-1)$-form on M: $\eta = r^*\omega$. Let $\iota : \partial M \to M$ be the inclusion: then $r \circ \iota = \mathrm{id}_{\partial M}$. Then

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} (r \circ \iota)^* \omega = \int_{\partial M} \iota^* r^* \omega \\
&= \int_{\partial M} \iota^* \eta = \int_M d\eta \\
&= \int_M dr^*\omega = \int_M r^* d\omega \\
&= \int_M 0 = 0.
\end{aligned}$$

But this contradicts the fact that ω is an orientation form.

Problem. (S13:09) Let $F : M \to N$ be a finite covering map between closed manifolds. Either prove or find counterexamples to the following questions.

(a) Do M and N have the same fundamental groups?

(b) Do M and N have the same de Rham cohomology groups?

(c) When M is simply connected, do M and N have the same singular homology groups?

Solution.
The same example works for all three parts: $S^2 \to \mathbb{RP}^2$.

16 Fall 2013

Problem. (F13:02) Let M be a connected compact manifold with boundary ∂M. Show that M does not retract onto ∂M.

Solution. We will show there is no smooth retraction. See also S13:05 for the orientable case using de Rham cohomology. Suppose $r : M \to \partial M$ is a smooth retract. By Sard's theorem, almost every point of ∂M is a regular value of r. If $p \in \partial M$ is a regular value, then $r^{-1}(p)$ is a submanifold of M of codimension $n-1$, i.e. dimension 1. Since M is compact, $r^{-1}(p)$ is also compact. By the classification of compact 1-manifolds, $r^{-1}(p)$ consists of a finite disjoint union of circles and closed intervals, and thus $\partial r^{-1}(p)$ has even cardinality. However, $\partial r^{-1}(p) \subset \partial M$, and $r^{-1}(p) \cap \partial M = \{p\}$. So the only way $\partial r^{-1}(p)$ can have even cardinality is if $r^{-1}(p)$ has empty boundary. But certainly p must be a boundary point of $r^{-1}(p)$, contradiction.

Problem. (F13:03) Let $M, N \subset \mathbb{R}^{p+1}$ be compact smooth oriented submanifolds of dimensions m and n, respectively, such that $m+n = p$. Suppose that $M \cap N = \varnothing$. Consider the map

$$\lambda : M \times N \to S^p : \lambda(x,y) = \frac{x-y}{|x-y|}.$$

The degree of λ is known as the *linking number* $l(M,N)$.

(a) Show that $l(M,N) = (-1)^{(m+1)(n+1)} l(N,M)$.

(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M,N) = 0$. (May require M to be compact and $\partial N = \varnothing$.)

Solution.

(a) Define the map

$$A : M \times N \to N \times M : A(x,y) = (y,x).$$

Define

$$\lambda_1 : M \times N \to S^p : \lambda_1(x,y) = \frac{x-y}{|x-y|}$$

and

$$\lambda_2 : N \times M \to S^p : \lambda_2(y,x) = \frac{y-x}{|y-x|}.$$

Then

$$\lambda_1(x,y) = -\lambda_2(A(x,y)) = r \circ \lambda_2 \circ A(x,y),$$

where $r : \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ is given by $r(z) = -z$. Since degrees are multiplicative across compositions,

$$\deg(\lambda_1) = \deg(r)\deg(\lambda_2)\deg(A).$$

$\deg(r) = (-1)^{p+1} = (-1)^{m+n+1}$, because r is the composition of $p+1$ reflections. As for A, note that A is a diffeomorphism. Fix $(y,x) \in N \times M$, so that

$$\deg A = \operatorname{sgn}(dA_{(x,y)}).$$

If $\mathcal{B}$ is an ordered basis of T_xM, and $\mathcal{B}'$ is an ordered basis of T_yN, then $dA_{(x,y)}$ sends the concatenated basis $(\mathcal{B},\mathcal{B}')$ to $(\mathcal{B}',\mathcal{B})$. Therefore the sign of $dA_{(x,y)}$ is $(-1)^{mn}$. Therefore

$$\deg(\lambda_1) = (-1)^{mn+1}\deg(\lambda_2).$$

Since $\deg(\lambda_1) = l(M,N)$ and $\deg(\lambda_2) = l(N,M)$, we have

$$l(M,N) = (-1)^{mn+m+n+1} l(N,M) = (-1)^{(m+1)(n+1)} l(N,M).$$

(b) If $\partial N = \varnothing$, then $\partial(W \times N) = M \times N$. Note that if $M = \partial W$, then λ extends to $W \times N$ by the same formula, because $W \cap N = \varnothing$. Since degrees are defined as intersection numbers, and intersection numbers are zero when a map extends from the boundary to the whole manifold, it follows that $\deg(\lambda) = l(M,N) = 0$.

Problem. (F13:04) Let ω be a 1-form on a connected manifold M. Show that ω is exact if and only if for all piecewise smooth closed curves $c : S^1 \to M$ it follows that $\int_c \omega = 0$.

Solution. See S13:02.

Problem. (F13:05) Let ω be a smooth nowhere vanishing 1-form on a 3-manifold M^3.

(a) Show that $\ker\omega$ is an integrable distribution on M if and only if $\omega\wedge d\omega = 0$.

(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

Solution.

(a) Suppose $\ker\omega$ is an integrable distribution. The ideal of forms vanishing on $\ker\omega$, $I(\ker\omega)$, is generated by the 1-form ω in the sense that for all $\eta \in I(\ker\omega)$, there is a form θ such that $\eta = \theta\wedge\omega$. By Frobenius, $I(\ker\omega)$ is also a differential ideal, so $d\omega \in I(\ker\omega)$. Then $d\omega = \theta\wedge\omega$ for some form θ. Then $\omega\wedge d\omega = \omega\wedge\theta\wedge\omega = 0$.

Suppose $\omega\wedge d\omega = 0$. We check that $\ker\omega$ is involutive. Let X,Y be local vector fields in $\ker\omega$: we must show that $[X,Y] \in \ker\omega$ as well. We may assume without loss of generality that X and Y are linearly independent. We have
$$d\omega(X,Y) = -\frac{1}{2}\omega([X,Y]).$$
Now, let Z be another local vector field such that X,Y,Z are linearly independent; one exists because the tangent spaces are 3-dimensional. We then compute:
$$\begin{aligned}
0 &= \omega\wedge d\omega(X,Y,Z)\\
&= \omega(X)d\omega(Y,Z) - \omega(Y)d\omega(Z,X) + \omega(Z)d\omega(X,Y)\\
&= \omega(Z)d\omega(X,Y).
\end{aligned}$$
But $\omega(Z)\neq 0$, since $\ker\omega$ is 2-dimensional and X,Y,Z are independent. Therefore $d\omega(X,Y) = 0$. Therefore $\omega([X,Y]) = 0$, and hence $\ker\omega$ is involutive.

(b) Take $M = \mathbb{R}^3$ and the plane field spanned by $X = \partial_x + y\partial_z$, $Y = \partial_y$. This distribution fails to be involutive: one finds that $[X,Y] = -\partial_z$, which is not in the span of X and Y.

Problem. (F13:07) Let $M = T^2 - D^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.

Solution. Let $\tilde{M}$ be such a surface. Since $\tilde{M}$ is a finite covering of a compact space, it follows that $\tilde{M}$ is compact. M is the genus 1 compact orientable surface with 1 hole, so $\chi(M) = 2 - 2 \cdot 1 - 1 = -1$. Then since $\tilde{M}$ is a 3-sheeted cover, we have

$$\chi(\tilde{M}) = -3\chi(M) = -3.$$

Finally, $\tilde{M}$ inherits the orientability of M. Choosing among compact orientable surfaces with Euler characteristic -3, by classification of surfaces we have the following choices for $\Sigma_{g,n}$ (the compact orientable surface of genus g and n holes):

$$\Sigma_{2,1}, \Sigma_{1,3}, \Sigma_{0,5}.$$

From here, we can rule out $\Sigma_{0,5}$. This is because the covering map must send boundary points to boundary points (since it is a local homeomorphism, and boundary points have neighborhoods homeomorphic to a half-plane while non-boundary points do not). If $\Sigma_{0,5}$ were to cover $\Sigma_{1,1}$, then the five boundary circles of $\Sigma_{0,5}$ would necessarily map to the one boundary circle of $\Sigma_{1,1}$, and so on the boundary the covering would be at least 5-fold. So our list of candidates is reduced to

$$\Sigma_{2,1}, \Sigma_{1,3}.$$

$\Sigma_{1,3}$ can be realized as a covering fairly easily: basically take three copies of the standard CW structure for $\Sigma_{1,1}$ and glue them side by side. For $\Sigma_{2,1}$, one can construct an irregular cover corresponding to a non-normal index 3 subgroup of $\mathbb{Z} * \mathbb{Z}$, and examining the corresponding homomorphism into S_3 induced by its action on the fiber of the basepoint. (See `http://math.stackexchange.com/q/1901523/98602` for details.)

Problem. (F13:09) Let $H \subset S^3$ be the Hopf link. Compute the fundamental groups and homology of the complement $S^3 - H$.

Solution. First, regard S^3 as the one-point compactification of $\mathbb{R}^3$. Consider $\mathbb{R}^3 - S^1$. This space is the union of disjoint tori, forming increasing shells around $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, and the z-axis. Post-compactification, the z-axis compactifies to the other circle S^1 in H. Therefore $S^3 - S^1$ can be thought of as $\mathbb{R}^3 - \{z\text{-axis}\}$, and $S^3 - H$ can be thought of as $\mathbb{R}^3 - \{z\text{-axis}\} - S^1$. This deformation retracts to two disjoint circles. Consequently the fundamental group is $\mathbb{Z}$ (regardless of choice of basepoint) and the homology is $\mathbb{Z}^2$ in dimensions 0 and 1, and 0 for all other dimensions.

Problem. (F13:10) Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions, with relations $i^2 = j^2 = -1$, $ij = ji = -k$. The multiplicative group $\mathbb{H}^* = \mathbb{H} - \{0\}$ acts on $\mathbb{H}^n - \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^*$ is called the quaternionic projective space. Calculate its homology groups.

Solution. We claim that $\mathbb{HP}^{n-1}$ has a cell decomposition with one cell in every fourth dimension, up to dimension $4n-4$. From here it follows immediately that $H_{4k}(\mathbb{HP}^{n-1};\mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq k \leq n-1$, and $H_j(\mathbb{HP}^{n-1};\mathbb{Z}) = 0$ otherwise. To see that we have this cell structure, we will adapt the description of the cell structure for $\mathbb{CP}^n$ from Hatcher. Note first that the quotient is equivalent to taking the quotient of $S^{4n-1} \subset \mathbb{H}^n \simeq \mathbb{R}^{4n}$ by the unit quaternions $h \in \mathbb{H}$, i.e. $h = (x,y,z,w)$ with $|h|^2 = |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1$. Now, consider

$$S^{4n-1}_{+} = \{(z,w) \in \mathbb{H}^{n-1} \oplus \mathbb{H} : (z,w) \in S^{4n-1}, w \in \mathbb{R}_{\geq 0}\} \simeq D^{4n-1}.$$

By $w \in \mathbb{R}$ we mean w is a quaternion that is purely real, i.e. $w = w + 0i + 0j + 0k$. Now, it is a fact that if $h \in \mathbb{H} - \{0\}$, then there exists a unit quaternion u such that $uh \in \mathbb{R}_{\geq 0}$ (by unique polar decomposition of quaternions, which the exam writers apparently expect to be common knowledge). Also, all elements of $\partial S^{4n-1}_{+} \simeq S^{4n-5}$ are related to one another by a unit quaternion, quaternions being a division algebra. Thus $\mathbb{HP}^n$ is formed by taking the quotient of D^{4n-1} by the unit quaternions on the boundary, and the quotient of the boundary S^{4n-5} by unit quaternions defines (after the obvious induction) an attaching map to $\mathbb{HP}^{n-2}$.

17 Spring 2014

Problem. (S14:01)
Let $\Gamma \subset \mathbb{R}^2$ be the graph of of the function $y = |x|$.

(a) Construct a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ whose image is Γ.

(b) Can f be an immersion?

Solution.

(a) Set
$$f(t) = e^{-\frac{1}{t^2}}(t, |t|).$$
Then f is smooth. The only point of concern is $t = 0$, but here it suffices to observe that $|t|e^{-\frac{1}{t^2}}$ is smooth.

(b) f cannot be an immersion. Write $f = (g,h)$. If f were an immersion, then $df = (g',h')$ would have to be nonvanishing. Clearly h cannot be monotone, because the graph of h is that of $|x|$ and this has a local minimum, so h' cannot be nonvanishing. Also g' cannot be nonvanishing; because then $g'(0)$ would be nonzero, so g would be a local diffeomorphism about 0. Then $h \circ g^{-1}$ must be smooth in a neighborhood of 0. But $h(g^{-1}(x)) = |x|$, which cannot be smooth.

Problem. (S14:02) Let W be a smooth manifold with boundary, and $f : \partial W \to \mathbb{R}^n$ a smooth map, for some $n \geq 1$. Show that there exists a smooth map $F : W \to \mathbb{R}^n$ such that $F \mid_{\partial W} = f$.

Solution.
By Whitney embedding, we may take W to be embedded in $\mathbb{R}^N$ for some N sufficiently large. Give $\mathbb{R}^N$ and W the Riemannian structure induced by the Euclidean inner product. Let NW denote the normal bundle of W. By the tubular neighborhood theorem there exists a neighborhood $U \subset \mathbb{R}^N$ of ∂W that is the diffeomorphic image under the map $E : N\partial W \to \mathbb{R}^N : (p,v) \mapsto p + v$ of a set of the form

$$V = \{(p,v) \in N\partial W : |v| < \delta(p)\},$$

where $\delta : \partial W \to (0,\infty)$ is some positive continuous function. Moreover, by choosing δ to be a sufficiently small function, we may assume that each $q \in U$ has a nearest point in ∂W, and the map $q \mapsto F(q)$ sending q to this nearest point is a submersion $F : U \to \partial W$. Now take a smooth compactly supported function $g : U \to \mathbb{R}$ that is equal to 1 on ∂W (one can be constructed by covering ∂W with balls and using a partition of unity argument). Define the extension of f to U by

$$f(q) = f(F(q))g(q).$$

Since $F(q) = q$ for $q \in \partial W$, and $g(q) = 1$ on ∂W, this does indeed define a smooth extension. Since g has compact support, we can now extend f by 0 to $W \setminus U$, and this produces a smooth extension of f to W.

Problem. (S14:03) Determine the values of $n \geq 0$ for which the antipodal map $S^n \to S^n : x \mapsto -x$ is isotopic to the identity.

Solution. The antipodal map on S^n can be achieved by $n+1$ reflections, along one coordinate direction at a time. Each reflection is a map of degree -1 since it is an orientation-reversing diffeomorphism. Therefore the antipodal map has degree

$(-1)^{n+1}$. By the Hopf degree theorem, this can only be homotopic to the identity if n is odd. So at the minimum we require that n is odd. This is also sufficient, because an even number of reflections can be obtained by rotations, which are isotopies.

Problem. (S14:04) Let $\omega_1, \dots, \omega_k$ be 1-forms on a smooth n-manifold M. Show that $\{\omega_i\}$ are linearly independent if and only if

$$\omega_1 \wedge \cdots \wedge \omega_k \neq 0.$$

Solution. See S15:03 for a related problem.

Assume that $\{\omega_i\}$ are linearly independent. Let $p \in M$, and complete $\{\omega_i\}$ to a basis $\omega_1, \dots, \omega_n$ of T_p^*M. Then the k-fold wedge products $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$, $i_1 < \cdots < i_k$, form a basis of the k-th exterior power $\bigwedge^k(T_p^*M)$. Then clearly we must have $(\omega_1 \wedge \cdots \wedge \omega_k)_p \neq 0$. Since $p \in M$ was arbitrary, we in fact see that $\omega_1 \wedge \cdots \wedge \omega_k$ is nonvanishing.

Conversely, suppose $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$. Let $p \in M$ with $(\omega_1 \wedge \cdots \wedge \omega_k)_p \neq 0$. Suppose $a_1, \dots, a_k \in \mathbb{R}$ such that

$$0 = \sum_{i=1}^{k} a_i (\omega_i)_p.$$

Then wedging with

$$\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k$$

(where the hat represents the omission of the corresponding 1-form) we find that

$$0 = (-1)^{i-1} a_i (\omega_1 \wedge \cdots \wedge \omega_k)_p.$$

for all $i = 1, \dots, k$. Since $(\omega_1 \wedge \cdots \wedge \omega_k)_p \neq 0$ we conclude that $a_i = 0$.

Problem. (S14:05) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M that represents the Poincaré dual of S.

Solution. See F15:04 for essentially the same problem.

Problem. (S14:07) Let $X = S^1 \vee S^1$. Give an example of an irregular covering space $\tilde{X} \to X$.

Solution. This amounts to finding a non-normal subgroup of $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. One such example is the subgroup of $\langle a, b \rangle$ generated by ab; this does not contain the conjugate

$$ba = (ba^{-1})(ab)(b^{-1}a) = (ba^{-1})(ab)(ba^{-1})^{-1}.$$

The connected covering space corresponding to this subgroup will be irregular. See Hatcher p.58 ex. (13) for a picture.

Problem. (S14:08) For $n \geq 2$, let X_n be the space obtained from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations. This produces a cell decomposition of X.

(a) Write down the associated cellular chain complex.

(b) Show that X_n is a surface, and find its genus.

Solution.

(a) As can be verified from drawing a few examples, the edge identifications lead to a cell structure with n edges, 1 face, and either 1 vertex if n is even, or 2 vertices if n is odd. The chain groups are free abelian on the basis of cells in each dimension. Thus for the cellular chain complex we have two cases:

1. If n is even, then we have one 0-cell v, n 1-cells $e_1, \ldots, e_n$, and one 2-cell f. The chain complex is

$$0 \longrightarrow \underset{2}{\mathbb{Z}\langle f \rangle} \xrightarrow{\partial_2} \underset{1}{\mathbb{Z}\langle e_1, \ldots, e_n \rangle} \xrightarrow{\partial_1} \underset{0}{\mathbb{Z}\langle v \rangle} \longrightarrow 0$$

where the cellular boundary maps are as follows. For ∂_2,

$$\partial_2(f) = e_1 + \cdots + e_n - e_1 - \cdots - e_n = 0$$

according to the word $e_1 \cdots e_n e_1^{-1} \cdots e_n^{-1}$ that defines the attaching map. For ∂_1,

$$\partial_1(e_i) = v - v = 0.$$

2. If n is odd, then now we have two 0-cells v and w, and the remaining data is the same. The chain complex is

$$0 \longrightarrow \underset{2}{\mathbb{Z}\langle f \rangle} \xrightarrow{\partial_2} \underset{1}{\mathbb{Z}\langle e_1, \ldots, e_n \rangle} \xrightarrow{\partial_1} \underset{0}{\mathbb{Z}\langle v, w \rangle} \longrightarrow 0$$

We still have

$$\partial_2(f) = 0,$$

but for ∂_1 we have

$$\partial_1(e_i) = v - w.$$

(b) The standard polygonal representation of a surface S is as the $2n$-gon with edges identified according to either the word $[e_1, e_2][e_3, e_4]\cdots[e_{n-1}, e_n]$ if S is orientable, where $[e_i, e_j] = e_i e_j e_i^{-1} e_j^{-1}$ is the commutator word, and $e_1^2 e_2^2 \cdots e_n^2$ if S is nonorientable. For X_n, the edge identifications give us instead the word $e_1 \cdots e_n e_1^{-1} \cdots e_n^{-1}$. We must show that this is equivalent to either of the two standard presentations. This uses techniques in the proof of classification of surfaces, which we will not prove in this limited space, but we will at least state the relevant results. (See any proof of classification of closed orientable surfaces for rigorous proofs of the details; for example, *Algebraic Topology: A First Course* by W. Fulton, Chapter 17.)

The basic rule is that if we have two letters α and β in the word so that they appear in a sequence

$$\cdots\alpha\cdots\beta\cdots\alpha^{-1}\cdots\beta^{-1}\cdots,$$

then by a sequence of cuts and new edge relabelings there is an equivalent word with the same number of letters (by which we mean a word whose edge identifications produce a homeomorphic space) with $\alpha, \beta, \alpha^{-1}, \beta^{-1}$ deleted, an intact commutator $[\gamma, \delta]$ added, and without disturbing other such commutator sequences. Thus in the case of n even, we can successively match pairs of distant commutator sequences until we are brought to the standard presentation of the genus $\frac{n}{2}$ orientable surface. For n odd, this results in one mismatched word of the form aa^{-1}, which cancels itself, and thus in this case we can delete the corresponding edge from the polygonal representation. Thus in this case we have the genus $\frac{n-1}{2}$ orientable surface.

If we do not wish to go through this procedure, there is a easy way to find the genus, provided we are willing to assume that X_n is indeed a surface. Taking alternating sums of the number of k-cells, we find that $\chi(X_n)$ is $1 - n + 1 = 2 - n$ when n is even, and $2 - n + 1 = 3 - n$ when n is odd. Since the word for the polygonal representation always introduces a letter a with its inverse a^{-1}, we know that the resulting surface will be orientable. Since all edges are identified, the surface will also be closed. The closed orientable surface of genus g has Euler characteristic $2-2g$. So for n even, we solve

$$2-2g=2-n$$

to find $g=\frac{n}{2}$, and for n odd we solve

$$2-2g=3-n$$

to find $g=\frac{n-1}{2}$, in agreement with our more involved approach.

Problem. (S14:09)

(a) Consider the space Y obtained from $S^2\times[0,1]$ by identifying $(x,0)$ with $(-x,0)$ and $(x,1)$ with $(-x,1)$ for all $x\in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{RP}^3\#\mathbb{RP}^3$.

(b) Show that $S^2\times S^1$ is a double cover of the connected sum $\mathbb{RP}^3\#\mathbb{RP}^3$.

Solution.

(a) Y consists of the union of $S^2\times(0,1)$ and $\mathbb{RP}^2\times\{0,1\}$. This is homeomorphic to $\mathbb{RP}^3\#\mathbb{RP}^3$ in the following way. The connected sum can be formed by deleting the top cells of each copy of $\mathbb{RP}^3$ (same as deleting a 3-ball), then attaching the underlying 2-skeleton (which is $\mathbb{RP}^2$) to the 3-cylinder $S^2\times[0,1]$ to each end along the map $S^2\to\mathbb{RP}^2$. This results in precisely the space Y as described above.

(b) Take S^1 as the unit circle in the complex plane. We give $\mathbb{RP}^3\#\mathbb{RP}^3$ the structure from part (a). The cover goes as follows: $S^2\times\{1\}$ covers $\mathbb{RP}^2\times\{0\}$ by the standard covering map $S^2\to\mathbb{RP}^2$. $S^2\times\{-1\}$ covers $\mathbb{RP}^2\times\{1\}$ in the same way. The remaining two arcs of S^1 contribute spaces which are homeomorphic to $S^2\times(0,1)$, each singly cover $S^2\times(0,1)$ (with the appropriate orientation on the arcs so that the map is continuous), so that each $S^2\times\{t\}\subset S^2\times(0,1)$ gets covered by two 2-spheres, one from each arc.

Problem. (S14:10) Let X be a topological space. Let SX, the suspension of X, be the space obtained from $X\times[0,1]$ by collapsing $X\times\{0\}$ to a point and $X\times\{1\}$ to another point. Determine the relationship between the homology of SX and of X.

Solution. Write $SX = A \cup B$, where

$$A = X \times [0, \frac{3}{4})/\sim,\ B = X \times (\frac{1}{4}, 1]/\sim.$$

Then A and B are cones over X, and in particular they deformation retract to a point. Their intersection is $X \times (\frac{1}{4}, \frac{3}{4})$, which deformation retracts to X. We thus obtain a long exact sequence

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(SX) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

which, after submitting our information about A, B, and $A \cap B$ becomes

$$\cdots \longrightarrow H_n(X) \longrightarrow 0 \longrightarrow H_n(SX) \longrightarrow H_{n-1}(X) \longrightarrow 0 \longrightarrow \cdots$$

In particular, we obtain isomorphisms $H_n(SX) \cong H_{n-1}(X)$ for all n, and the isomorphism is given by the Mayer-Vietoris connecting homomorphism.

18 Fall 2014

Problem. (F14:01) Let $f : M \to N$ be a proper immersion between connected (compact?) manifolds of the same dimension. Show that f is a covering map.

Solution. This is the stack of records theorem. We must show:

(a) *f is surjective.* Since f is an immersion, and $\dim M = \dim N$, f is a local diffeomorphism, and therefore an open map; thus $f(M)$ is open in N. Provided M is compact (or provided N is compact, which implies M is compact since f is proper), $f(M)$ is also closed. Since N is connected, $f(M) = N$.

(b) *Every $p \in N$ has a neighborhood $U \ni p$ such that $f^{-1}(U)$ is the disjoint union of open sets, each homeomorphic to U.* Let $p \in N$. Since $\{p\}$ is compact, and f is proper, $f^{-1}(p)$ is compact. Since f is a surjective local diffeomorphism, every point of N is a regular value of f, and therefore $f^{-1}(p)$ is a compact submanifold of M of codimension $\dim N = \dim M$; hence $f^{-1}(p)$ is a compact zero-dimensional submanifold, hence finite. Then for each $q \in f^{-1}(p)$, by the inverse function theorem there exist neighborhoods $V_q \ni q$ and $U_q \ni p$ such that the restriction $f : V_q \to U_q$ is a C^1 diffeomorphism. Moreover, since $f^{-1}(p)$ is finite, we may pick the V_q so that they are all disjoint, and take $U = \bigcap_{q \in f^{-1}(p)} f(V_q)$ so that each V_q is diffeomorphic through f to U. This completes the proof.

Problem. (F14:02)
Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.

(a) Show that $\mathbb{R}^n \setminus M^m$ is connected when $m \leq n-2$.

(b) When $m = n-1$, show that $\mathbb{R}^n \setminus M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

Solution.
See also F12:03.

(a) Let x, y be distinct points in $\mathbb{R}^n \setminus M^m$. Let $\gamma : I \to \mathbb{R}^n$ be a smooth path joining x and y. Since transversality is generic, γ is smoothly homotopic to another path $\tilde{\gamma}$ joining x and y that is transverse (both as a mapping and as a submanifold) to M. But then $\tilde{\gamma}$ is a path in $\mathbb{R}^n$ that does not intersect M. For suppose for contradiction that $\tilde{\gamma}(t) = p$ is such an intersection. Then by transversality, we have
$$d\tilde{\gamma}_t(T_t\mathbb{R}) + T_pM = \mathbb{R}^n.$$
But $T_t\mathbb{R}$ is one-dimensional, and M is m-dimensional; so the largest dimension the above vector space sum can have is $m+1 \leq n-1 < n$, contradiction. Therefore x and y can be joined by a path that does not intersect M, proving that M is path connected.

(b) I choose not to do this part on the grounds that the Jordan-Brouwer separation theorem is not a qual problem, but rather an exercise in sadism by the qual committee. An outline of the proof is in Guillemin & Pollack.

Problem. (F14:03)
Let ω be an n-form on a closed connected non-orientable smooth manifold M and let $\pi : \mathcal{O} \to M$ be the orientation cover.

(a) Show that $\pi^*\omega$ is exact.

(b) Show that ω is exact.

Solution.

(a) Since $\mathcal{O}$ is connected, closed, and orientable, its top cohomology is $H^n_{dR}(\mathcal{O}) \cong \mathbb{R}$, with this isomorphism going as follows:
$$[\theta] \longleftrightarrow \int_{\mathcal{O}} \theta \in \mathbb{R}.$$

In particular an n-form is zero in cohomology, i.e. exact, if and only if it integrates to 0 over M. So we claim that $\int_{\mathcal{O}} \pi^* \omega = 0$.

Consider the group of covering transformations $\{\mathrm{id}, F\}$ where F is the unique non-identity covering transformation. We observe that F is orientation-reversing. For suppose F were orientation-preserving. Then we could define an orientation on M, by first choosing an orientation on $\mathcal{O}$, and using the action of π to carry the orientation to M: for each $p \in f^{-1}(q)$, define the orientation at q to be the one induced by the local diffeomorphism $T_p\mathcal{O} \to T_qM$. This gives a well-defined orientation because F is orientation-preserving, so the induced orientation does not depend on the choice of point in the fiber.

By definition of degree, we have

$$\int_{\mathcal{O}} F^* \pi^* \omega = \deg F \int_{\mathcal{O}} \pi^* \omega = -\int_{\mathcal{O}} \pi^* \omega,$$

where $\deg F = -1$ because F is an orientation-reversing diffeomorphism. On the other hand, since F is a covering transformation, $\pi \circ F = \pi$, and therefore $F^* \pi^* = \pi^*$. Therefore

$$\int_{\mathcal{O}} \pi^* \omega = \int_{\mathcal{O}} F^* \pi^* \omega = -\int_{\mathcal{O}} \pi^* \omega,$$

and the claim is proved.

(b) Same argument as S10:05. Show that the covering map induces a pullback that is injective on forms. Average the form obtained in (a) over the action of the deck transformations and show that this can be used to define a form on M. Show that this form is exact by leveraging the injectivity of the pullback.

Problem. (F14:04) Show that for $n \geq 1$, any smooth map $f : S^{n-1} \to S^{n-1}$ has a smooth extension $F : D^n \to D^n$.

Solution. Let $\varphi : [0, \infty) \to [0, 1]$ be a smooth function supported in $[\frac{1}{2}, \frac{3}{2}]$ which is 1 on $[\frac{3}{4}, \frac{5}{4}]$. (One can be constructed by convolving the indicator function of $[\frac{3}{4}, \frac{5}{4}]$ with an appropriate smooth compactly supported mollifier.) Define $\Phi : \mathbb{R}^n \to [0, 1]$ as the radially symmetric function with $\Phi(x) = \varphi(|x|)$; then Φ is supported in the annulus $\{\frac{1}{2} \leq |x| \leq \frac{3}{2}\}$ and is 1 on the annulus $\{\frac{3}{4} \leq |x| \leq \frac{5}{4}\}$. Define $F : D^n \to D^n$ by

$$F(x) = \begin{cases} \Phi(x) f(x/|x|) & 0 < |x| \leq 1, \\ 0 & x = 0. \end{cases}$$

Then this is a smooth extension of f.

Problem. (F14:05) Let M be a smooth manifold and ω a nowhere-vanishing 1-form on M. Show that ω is locally proportional to the differential of a function (i.e. around each $p \in M$ there exists a neighborhood $U \ni p$ and functions $f, \lambda : U \to \mathbb{R}$ such that $\omega = \lambda df$ on U) if and only if $\omega \wedge d\omega = 0$.

Solution. Suppose ω is locally proportional to the differential of f at p. Then since the exterior derivative can be computed purely locally, we have

$$d\omega = d\lambda \wedge df.$$

Then

$$\omega \wedge d\omega = \lambda df \wedge d\lambda \wedge df = 0.$$

Conversely, suppose $\omega \wedge d\omega = 0$. Then $\ker \omega$ is an involutive distribution of codimension 1; see F13:05. Therefore at every point $p \in M$, there is an immersed integral submanifold N through p, that is, $T_pN = \ker \omega_p$. An immersed submanifold of codimension 1 can locally be expressed as the zero locus of a function, so there is a neighborhood $U \ni p$ so that $N \cap U = \{q \in U : f(q) = 0\}$. Then $df_q(X) = 0$ for any $X \in T_qN = \ker \omega_q$, so $\ker \omega_q = \ker df_q$. Thus it follows that $df_q = \lambda(q)\omega_q$ for some $\lambda(q) \neq 0$, which proves the claim.

Problem. (F14:06) Recall that the rank of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

Solution. See S15:01.

Problem. (F14:08)
Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$.

(a) Determine the fundamental group of X.

(b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

Solution.
Here is a solution that handles both parts at once. X is the space obtained by taking two Möbius bands and attaching them to each other by a homeomorphism along their boundaries. This is homotopy equivalent to the space obtained by

taking a cylinder, equivalently a sphere with two holes punched out, and attaching a Möbius band to each boundary circle along the boundary of the Möbius band. This is precisely the description of the closed nonorientable surface of genus 2 from classification of surfaces. Therefore the fundamental group is $\langle a, b \mid a^2b^2\rangle$ and X cannot be homotopy equivalent to a compact orientable surface of genus g, because the homology is invariant under homotopy equivalence. X has first homology $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the closed surface of genus g has $\mathbb{Z}^{2g}$ instead.

Problem. (F14:10) Let D be the unit disk in the complex plane, let S^1 be the unit circle in the complex plane. Consider the 2-torus T^2 and two discs D_1 and D_2. Let X be the quotient of the disjoint union $T^2 \coprod D_1 \coprod D_2$ by the equivalence relations

$$e^{i\theta} \sim (e^{ip\theta}, 1),\ e^{i\phi} \sim (1, e^{iq\phi})$$

where $e^{i\theta} \in D_1$, $e^{i\phi} \in D_2$, and p, q are integers > 1. Find the homology groups of X.

Solution. X has the following CW structure:

1. One 0-cell v.
2. Two 1-cells a, b attached to v along their endpoints.
3. Three 2-cells α, β, γ attached to the 1-cells along the following words:

$$\alpha:\ aba^{-1}b^{-1};\ \beta: a^p;\ \gamma: b^q.$$

This gives rise to the following cellular chain complex:

$$0 \longrightarrow \mathbb{Z}\langle \alpha, \beta, \gamma\rangle \cong \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}\langle a, b\rangle \cong \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}\langle v\rangle \cong \mathbb{Z} \longrightarrow 0$$

where the cellular boundary maps are given as follows:

$$\begin{aligned}
\partial_2(\alpha) &= a + b - a - b = 0;\\
\partial_2(\beta) &= pa;\\
\partial_2(\gamma) &= qb;\\
\partial_1(a) &= \partial_1(b) = v - v = 0.
\end{aligned}$$

From here we see that $H_0(X) \cong \mathbb{Z}$ $\partial_3 : \mathbb{Z}^3 \to \mathbb{Z}^2$ is given by the matrix

$$\begin{bmatrix} 0 & p & 0 \\ 0 & 0 & q \end{bmatrix}$$

and so im ∂_3 has rank 2, $\ker \partial_3$ has rank 1. So

$$H_1(X) = \frac{\ker \partial_1}{\text{im } \partial_2} \cong \frac{\mathbb{Z}^2}{\mathbb{Z}^2} = 0,$$

$$H_2(X) = \ker \partial_2 \cong \mathbb{Z}.$$

19 Spring 2015

Problem. (S15:01) Let $M(n,m,k) \subset M(n,m)$ denote the space of $n \times m$ matrices of rank k. Show that $M(n,m,k)$ is a smooth manifold of dimension $nm - (n-k)(m-k)$.

Solution. Let Z denote the set of all block matrices of the form

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B is a $k \times k$ nonsingular matrix. Then Z is an open submanifold of $M(n,m)$, because $M(n,m) \simeq \mathbb{R}^{nm}$ and the nonsingularity of B is an open condition. Let $A \in Z$ be as above. Then right multiplying by the $m \times m$ block matrix

$$X = \begin{bmatrix} \text{id}_{k\times k} & -B^{-1}C \\ 0 & \text{id}_{(m-k)\times(m-k)} \end{bmatrix}$$

we obtain the $n \times m$ block matrix

$$AX = \begin{bmatrix} B & 0 \\ D & -DB^{-1}C + E \end{bmatrix}$$

Since X is nonsingular, A has rank k if and only if AX has rank k, which is true if and only if $-DB^{-1}C + E = 0$.

Define the map

$$F : Z \to M(n-k.m-k) : \begin{bmatrix} B & C \\ D & E \end{bmatrix} \mapsto -DB^{-1}C + E.$$

F is clearly smooth. We claim that the zero matrix is a regular value of F. Let $Y \in M(n-k.,m-k)$, and consider the path

$$\gamma(t) = \begin{bmatrix} B & C \\ D & E + tY \end{bmatrix}.$$

γ is a smooth curve in Z with $\gamma'(0) = Y$. Then for any $A \in Z$,

$$dF_A \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} = \frac{d}{dt}\bigg|_{t=0} (F \circ \gamma) = \frac{d}{dt}\bigg|_{t=0} (-DB^{-1}C + E + tY) = Y.$$

Therefore $dF_A : T_A Z \to T_{F(A)} M(n-k, m-k) \cong M(n-k, m-k)$ is surjective, regardless of whether $F(A) = 0$. We thus conclude that 0 is a regular value of F, and hence $F^{-1}(0)$ is a codimension $(n-k)(m-k)$ submanifold of Z, and therefore a submanifold of $M(n,m)$ of dimension $nm - (n-k)(m-k)$.

Finally, we note that every element of $M(n,m,k)$ is related through elementary row operations to a matrix in Z. We form a cover of $M(n,m,k)$ by sets of the form $E(Z)$, where E is an elementary row matrix. Noting that left multiplication by an elementary row matrix E is a smooth operation (work in the usual Euclidean coordinates), and the transition maps are also smooth for the same reasons, this induces the smooth structure on $M(n,m,k)$.

Problem. (S15:02) Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that N can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f : M \to \mathbb{R}$, if and only if there is a vector field X on M that is transverse to N.

Solution. Let g be a Riemannian metric on M. First assume $N = f^{-1}(0)$ as above. Define the gradient vector field ∇f on M as the dual vector field to the 1-form df: i.e. $df(\cdot) = g(\nabla f, \cdot)$. Note that since f is constant on N, $df \equiv 0$ on $TN \subset TM$. Therefore $\nabla f \perp TN$. But note also that $(\nabla f)_p \neq 0$ for all $p \in N$, because 0 is a regular value of f and therefore df cannot vanish on N. Since N is codimension 1, it follows that $T_pN \oplus \operatorname{span}(\nabla f)_p = T_pM$ for all $p \in N$, so ∇f is transverse to N.

The converse assertion is actually false; see the following MSE question/answer (credit to Austin Christian for recognizing an opportunity to give up intelligently).

Problem. (S15:03) Consider two collections of 1-forms $\omega_1, \ldots, \omega_k$ and $\phi_1, \ldots, \phi_k$ on an n-dimensional manifold M. Assume that

$$\omega_1 \wedge \cdots \wedge \omega_k = \phi_1 \wedge \cdots \wedge \phi_k$$

never vanishes on M. Show that there are smooth functions $f_{ij} : M \to \mathbb{R}$ suc that

$$\omega_i = \sum_{j=1}^{k} f_{ij} \phi_j, \ \ i = 1, \ldots, k.$$

Solution.
From S14:04 we know that $\{\omega_i\}$ forms a linearly independent set of 1-forms, and likewise for $\{\phi_i\}$. Complete $\{\phi_i\}$ to a $C^\infty(M)$-module basis $\phi_1, \dots, \phi_n$ of $\Omega^1(M)$, the space of smooth differential 1-forms on M. Then there exist smooth functions $f_{ij} : M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{n} f_{ij}\phi_j, \quad i = 1, \dots, k.$$

Then

$$\begin{aligned}
\omega_1 \wedge \cdots \wedge \omega_k &= \left(\sum_{j=1}^{n} f_{1j}\phi_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} f_{kj}\phi_j\right) \\
&= \sum_{1 \le j_1, \dots, j_k \le n} \left(\prod_{i=1}^{k} f_{ij_i}\right) \phi_{j_1} \wedge \cdots \wedge \phi_{j_k} \\
&= \phi_1 \wedge \cdots \wedge \phi_k.
\end{aligned}$$

The k-fold wedge products $\phi_{j_1} \wedge \cdots \wedge \phi_{j_k}$ form a basis of $\Omega^k(M)$. Therefore if $\{j_1, \dots, j_k\} \notin \{1, \dots, k\}$, then

$$\prod_{i=1}^{k} f_{ij_i} \equiv 0.$$

We conclude that $f_{ij} \equiv 0$ if $j \notin \{1, \dots, k\}$. Therefore

$$\omega_i = \sum_{j=1}^{k} f_{ij}\phi_j.$$

Problem. (S15:04) Consider a smooth map $F : \mathbb{RP}^n \to \mathbb{RP}^n$.

(a) When n is even, show that F has a fixed point.

(b) When n is odd, give an example where F does not have a fixed point.

Solution.

(a) We define a map $G : S^n \to S^n$ as follows: let D^n_+ be the closed upper hemisphere of S^n, obtained by taking the last coordinate positive, and D^n_- the closed lower hemisphere. For each $p \in \mathbb{RP}^n$ whose fiber does not lie in

$\partial D^n_+ \simeq S^{n-1}$, the fiber of p by the covering map $\pi : S^n \to \mathbb{RP}^n$ consists of two points $\{p_+, -p_+\}$, where $p_+ \in D^n_+$. We define G on $S^n \setminus \partial D^n_+$ by

$$G(p_+) = (F(p))_+,\ G(-p_+) = (F(p))_-.$$

We then extend G continuously to all of S^n to obtain a map $S^n \to S^n$; this can be done by applying uniqueness of path liftings. Therefore we have obtained a map $G : S^n \to S^n$ with the property that $\pi \circ G = F \circ \pi$. Now, since n is even, every map $H : S^n \to S^n$ has a point $x \in S^n$ with either $H(x) = x$ or $H(x) = -x$. For if $H(x) \neq x$ for all x, then

$$H_t(x) = \frac{(1-t)H(x) - tx}{|(1-t)H(x) - tx|}$$

provides a homotopy between H and $-\mathrm{id}_{S^n}$, while if $H(x) \neq -x$ for all x then

$$H_t(x) = \frac{(1-t)H(x) + tx}{|(1-t)H(x) + tx|}$$

is a homotopy between H and id_{S^n}. But if n is even, then $\deg(\mathrm{id}_{S^n}) = 1$ and $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1} = -1$, contradiction. Therefore G either has a fixed point or maps a point to its antipode. In either case, call such a point p: then

$$F(\pi(p)) = \pi(G(p)) = \pi(\pm p) = \pi(p).$$

Then $\pi(p)$ is the desired fixed point.

For a fancier way to do this proof, we can use cohomology rings. The integral cohomology ring of $\mathbb{RP}^n$, n even, is

$$H^*(\mathbb{RP}^n) \simeq \mathbb{Z}[x]/\langle 2x, x^{\frac{n}{2}+1} \rangle,$$

where the coefficient ring is $H^0(\mathbb{RP}^n)$, x is the non-identity element of $H^2(\mathbb{RP}^n) \simeq \mathbb{Z}/2\mathbb{Z}$, and x^r is the non-identity element of $H^{2r}(\mathbb{RP}^n)$ for $1 \leq r \leq \frac{n}{2}$. In particular the cohomology ring is generated in H^2. If $F : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a map, then for $1 \leq r \leq \frac{n}{2}$ the trace of $F^* : H^{2r}(\mathbb{RP}^n) \to H^{2r}(\mathbb{RP}^n)$ is zero, because these cohomology groups are $\mathbb{Z}/2\mathbb{Z}$; and the trace of $F^* : H^k(\mathbb{RP}^n) \to H^k(\mathbb{RP}^n)$ is zero for all odd k, since these cohomology groups are zero. Therefore the Lefschetz number of F is

$$\tau(F) = \mathrm{tr}(F^* : H^0(\mathbb{RP}^n) \to H^0(\mathbb{RP}^n)) = [F^* x]^0 = (\pm 1)^0 = 1.$$

By the Lefschetz fixed point theorem it follows that F has a fixed point.

(b) For $n = 1$, $\mathbb{RP}^1 = S^1$, so an easy example is to take a rotation by a an angle that is a non-integer multiple of 2π on S^1.

Problem. (S15:05) Assume we have a codimension 1 distribution $\Delta \subset TM$.

(a) Show if the quotient bundle TM/Δ is trivial (or equivalently that there is a vector field on M that never lies in Δ), then there is a 1-form ω on M such that $\Delta = \ker \omega$ everywhere on M.

(b) Give an example where TM/Δ is not trivial.

(c) With ω_1 and ω_2 as in (a) show that $\omega_1 \wedge d\omega_1 = f^2 \omega_2 \wedge d\omega_2$ for a smooth function f that never vanishes.

(d) If ω is as in (a) and $\omega \wedge d\omega \neq 0$, show that Δ is not integrable.

Solution.

(a) Suppose TM/Δ is trivial, that is, $TM/\Delta = M \times \mathbb{R}$. Define a 1-form ω as follows: for $p \in M$ and $X_p \in T_pM$, let

$$\omega(X_p) = [X_p] \in T_pM/\Delta_p = \mathbb{R}.$$

Since the value of a form on vector fields is entirely determined pointwise, this gives rise to a smooth 1-form, and $\ker \omega = \Delta$ essentially by construction.

(b) ?

(c) Since Δ is a codimension 1 distribution equal to $\ker \omega_1$, the ideal $I(\Delta)$ of forms vanishing on Δ is generated by ω_1, in the sense that every k-form in $I(\Delta)$ can be expressed as

$$\eta = \theta \wedge \omega_1$$

for a unique $(k-1)$-form θ. In particular, $\omega_2 = f\omega_1$ for some smooth function f, and f is nonvanishing because $\ker \omega_2$ is a codimension 1 distribution: if $f(p) = 0$ for some p, then $(\omega_2)_p : T_pM \to \mathbb{R}$ would be zero, and hence $\ker(\omega_2)_p \subset T_pM$ would fail to be codimension 1. Then

$$\omega_2 \wedge d\omega_1 = (f\omega_1) \wedge d(f\omega_1) = f^2 \omega_1 \wedge d\omega_1 + f\omega_1 \wedge df \wedge \omega_1 = f^2 \omega_1 \wedge d\omega_1.$$

(d) See F13:05.

Problem. (S15:06) Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

be a 2-form defined on $\mathbb{R}^3 \setminus \{0\}$. If $i : S^2 \to \mathbb{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ maps $(x,y,z) \to (3x, 2y, 8z)$.

Solution. For the first integral, note that

$$i^*\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

We compute

$$di^*\omega = 3dx \wedge dy \wedge dz.$$

Let $B^3 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$. Then $S^2 = \partial B^3$. By Stokes' theorem,

$$\int_{S^2} i^*\omega = \int_{B^3} d(i^*\omega) = \int_{B^3} 3dx \wedge dy \wedge dz = 3\mathrm{Vol}(B^3).$$

For $j^*\omega$, note that j is an orientation-preserving diffeomorphism from S^2 and its interior to the ellipsoid

$$E = \{(x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{8}\right)^2 = 1\}$$

and its interior

$$V = \{(x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{8}\right)^2 \leq 1\}.$$

We compute

$$j^*\omega = 48\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{((3x)^2+(2y)^2+(8z)^2)^{\frac{3}{z}}} = \frac{48}{u(x,y,z)^{\frac{3}{2}}}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy),$$

where

$$u(x,y,z) = (3x)^2 + (2y)^2 + (8z)^2.$$

Then

$$\begin{aligned} dj^*\omega &= -\frac{72}{u(x,y,z)^{\frac{5}{2}}}(x\partial_x u + y\partial_y u + z\partial_z u)(dx \wedge dy \wedge dz) \\ &= \frac{-72(18x+8y+128z)}{((3x)^2+(2y)^2+(8z)^2)^{\frac{5}{2}}} dx \wedge dy \wedge dz. \end{aligned}$$

Therefore by Stokes' theorem, and the change of variables

$$F(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)) = (\frac{u}{3}, \frac{y}{2}, \frac{z}{8}),$$

we obtain

$$\int_{S^2} i^* j^* \omega = \int_{B^3} \frac{-72(18x+8y+128z)}{((3x)^2+(2y)^2+(8z)^2)^{\frac{5}{2}}}\, dxdydz$$
$$= \int_V -3\frac{(3u+2v+8w)}{(u^2+v^2+w^2)^{\frac{5}{2}}}\, dudvdw.$$

Now we note that the integrand is an odd function of each variable. Therefore by the reflection symmetries of the domain, it follows that

$$\int_{S^2} i^* j^* \omega = 0.$$

Problem. (S15:07) Define the de Rham cohomology groups $H^i_{dR}(M)$ of a manifold M and compute $H^i_{dR}(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}$, $i = 0, 1, \ldots$ directly from the definition.

Solution. Presumably the reader can define the de Rham groups. Since S^1 is 1-dimensional, there are no 2-forms, so it suffices to determine $H^0_{dR}(S^1)$ and $H^1_{dR}(S^1)$. We have the following complex of forms:

$$0 \longrightarrow \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \longrightarrow 0$$

$\Omega^0(S^1)$ consists of smooth functions on S^1, i.e. 1-periodic smooth functions. $\Omega^1(S^1)$ consists of 1-forms $fd\theta$, where f is a 1-periodic smooth function and $d\theta$ is the usual angular coordinate 1-form. If f is a 0-form, then $df = \frac{\partial f}{\partial \theta} d\theta$. Therefore $H^0_{dR}(S^1) = \ker d \subset \Omega^0(S^1)$ consists of those 1-periodic functions that are constant in θ, and hence is isomorphic to the vector space of real constants $\mathbb{R}$. im $d \subset \Omega^1(S^1)$ consists of forms $\frac{ptf}{\partial \theta} d\theta$. Note that if $\omega = fd\theta$, f periodic, then we can define a periodic function g with $dg = \omega$ by taking

$$g(\theta) = \int_0^\theta f(t)\, dt,$$

$\theta \in [0, 1)$, and then extending periodically; this always works, provided that f is not constant (because a constant c integrates to the function $c\theta$, which does not extend periodically). Therefore $H^1_{dR}(S^1) = \Omega^1(S^1)/\text{im } d$ consists of constant forms $\omega = cd\theta$, $c \in \mathbb{R}$, and thus is isomorphic to $\mathbb{R}$.

Problem. (S15:08) Let X be a CW complex consisting of one vertex p, 2 edges a and b attached to p along their boundaries, and 2 faces f_1 and f_2, attached along ab^2 and ba^2 respectively.

(a) Find $\pi_1(X)$. Is it a finite group?

(b) Compute the homology groups $H_i(X)$ of X.

Solution.

(a) The 1-skeleton of this CW complex is the wedge of 2 circles, oriented along a and b respectively. This gives us two generators a and b for the presentation of the group. Attaching f_1 along ab^2 introduces the relation ab^2, and attaching f_2 along ba^2 introduces the relation ba^2. Therefore

$$\pi_1(X) = \langle a, b \mid ab^2, ba^2 \rangle.$$

From the relations we have

$$ab^2 = e = ba^2,$$

so

$$b = a.$$

So in fact the group is generated by one element, and we have

$$e = ab^2 = a^3,$$

so

$$\pi_1(X) = \langle a \mid a^3 \rangle = \mathbb{Z}/3\mathbb{Z},$$

which is of course finite.

(b) X has one connected component, so $H_0(X) \cong \mathbb{Z}$. By the Hurewicz homomorphism $H_1(X) \cong \mathbb{Z}/3\mathbb{Z}$. It remains to determine $H_2(X)$, which we will do by cellular homology. The group $C_2(X)$ of cellular 2-chains is generated by f_1, f_2, while the group of 1-chains is generated by a, b. $H_2(X)$ is equal to the kernel of the cellular boundary map $\partial : C_2(X) \to C_1(X)$. The cellular boundary map in this case gives

$$\partial(f_1) = a + 2b,\ \partial(f_2) = 2a + b.$$

Therefore the cellular boundary can be regarded as a map $\partial : \mathbb{Z}^2 \to \mathbb{Z}^2$ represented by the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

This matrix has trivial kernel. Therefore $H_2(X) = 0$.

Problem. (S15:09) Let X, Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \coprod Y$ by identifying $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$\cdots \longrightarrow H_i(X) \xrightarrow{a} H_i(Y) \xrightarrow{b} H_i(Z) \xrightarrow{c} H_{i-1}(X) \longrightarrow \cdots$$

Also describe the maps a, b, c.

Solution. Consider the map of pairs

$$q : (X \times I, X \times \partial I) \to (Z, Y)$$

given by the quotient. For each pair we have long exact sequences for relative homology

$$\cdots \longrightarrow H_{i+1}(X \times I, X \times \partial I) \xrightarrow{\delta} H_i(X \times \partial I) \xrightarrow{i_*} H_i(X \times I) \xrightarrow{j_*} \cdots$$

$$\cdots \longrightarrow H_{i+1}(Z,Y) \xrightarrow{\delta} H_i(Y) \xrightarrow{i_*} H_i(Z) \xrightarrow{j_*} \cdots$$

where i_* are induced by inclusions, j_* are induced by quotients on the level of chain complexes (in the usual way for relative homology of pairs), and δ are the connecting homomorphisms arising from the long exact sequence from short exact sequence of chain complexes construction. Then the map of pairs induces the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(X \times I, X \times \partial I) \arrow[r, "\delta"] \arrow[d, "q_*"] & H_i(X \times \partial I) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_i(X \times I) \arrow[r, "j_*"] \arrow[d, "q_*"] & \cdots \\
\cdots \arrow[r] & H_{i+1}(Z,Y) \arrow[r, "\delta"] & H_i(Y) \arrow[r, "i_*"] & H_i(Z) \arrow[r, "j_*"] & \cdots
\end{tikzcd}$$

Note that $X \times \partial I = X \times \{0\} \cup X \times \{1\}$, and $X \times I$ retracts onto either $X \times \{0\}$ or $X \times \{1\}$. Therefore i_* in the upper row is surjective. By exactness, $j_* = 0$ in the upper row, and δ is an isomorphism onto its image in the upper row. im $\delta = \ker i_*$, and $\ker i_*$ consists of pairs $(\alpha, -\alpha)$ for $\alpha \in H_i(X)$. Therefore this kernel is isomorphic to $H_i(X)$, and the middle q_* takes $(\alpha, -\alpha)$ to $f_*\alpha - g_*\alpha$. The left q_* is an isomorphism, because q induces a homeomorphism of quotient spaces

$$\frac{X \times I}{X \times \partial I} \cong \frac{Z}{Y}.$$

Thus we may replace $H_{i+1}(Z,Y)$ with $H_{i+1}(X \times I, X \times \partial I)$, and this we may replace with $\ker i_* \cong H_i(X)$. This gives us the exact sequence

$$\cdots \longrightarrow H_i(X) \xrightarrow{F} H_i(Y) \xrightarrow{i_*} H_i(Z) \xrightarrow{j_*} \cdots$$

Here i_* and j_* arise as before from the long exact sequence in relative homology. F arises in the following way:

$$\begin{tikzcd}
H_{i+1}(X \times I, X \times \partial I) \arrow[r, "\delta{,}\cong"] \arrow[d, "q_*{,}\cong"] & \ker i_* \cong H_i(X) \arrow[d, "f_* - g_*"] \\
H_{i+1}(Z,Y) \arrow[r, "\delta"] & H_i(Z)
\end{tikzcd}$$

The diagram commutes, so we may think of F as $(f_* - g_*)\delta q_*^{-1}$.

Problem. (S15:10) Let $n \geq 0$ be an integer. Let M be a compact (closed?), orientable smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Solution. By the de Rham theorem, we may identify $H^{2n+1}(M;\mathbb{R})$ canonically with $H^{2n+1}_{dR}(M)$. We define a bilinear form on $H^{2n+1}_{dR}(M)$ in the following way: given $[\omega], [\eta] \in H^{2n+1}_{dR}(M)$, and $\omega \in [\omega]$, $\eta \in [\eta]$ closed representatives,

$$([\omega],[\eta]) = \int_M \omega \wedge \eta.$$

(This map is well-defined, which is essentially the statement of Poincaré duality.) This is a skew-symmetric form: $([\omega],[\eta]) = -([\eta],[\omega])$. Then the linear map

$$[\omega] \mapsto ([\eta] \mapsto ([\omega],[\eta]))$$

is skew-symmetric, hence of even rank. But this is also precisely the isomorphism given by Poincaré duality, $H^{2n+1}_{dR}(M) \to H^{2n+1}(M;\mathbb{R})$. Therefore $\dim H^{2n+1}_{dR}(M)$ is even.

20 Fall 2015

Problem. (F15:01) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that $SL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$ is a smooth submanifold of $M_n(\mathbb{R})$.

(b) Show that $SL(n,\mathbb{R})$ has trivial Euler characteristic.

Solution.

(a) We show that the map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ has 1 as a regular value. Since $M_n(\mathbb{R})$ is diffeomorphic to n^2-dimensional Euclidean space, we may identify $T_A M_n(\mathbb{R}) \cong M_n(\mathbb{R})$. Then note that for any $A \in M_n(\mathbb{R})$,

$$d(\det)_A(A) = \lim_{t\to 0} \frac{\det(A+tA) - \det(A)}{t} = \lim_{t\to 0} \frac{(1+t)^n - 1}{t} \det(A) = n \det(A).$$

Therefore if $\det(A) \neq 0$, then $d(\det)_A : M_n(\mathbb{R}) \to \mathbb{R}$ is nonzero, and therefore of full rank, and hence any nonzero real number is a regular value of det. In particular 1 is a regular value of det.

(b) For any compact connected Lie group G, we can perform the following proof to show that the Euler characteristic is zero. Let $X \in T_e G$ be a nonzero vector field at the identity. We can produce a nonvanishing tangent vector field on G by using the left action of G on itself to translate X to $g_* X \in T_g G$. Then by the Poincaré-Hopf theorem, it follows that $\chi(G) = 0$. To adapt this to $SL(n,\mathbb{R})$ (which is not compact), it is enough to note that $SL(n,\mathbb{R})$ retracts to $SO(n)$, which is a compact Lie group.

Another proof can be given in the same vein by the Lefschetz fixed point theorem; in this case, we can use the fact that the left action on a compact Lie group G gives us diffeomorphisms with no fixed points. Therefore if $g : G \to G$ is the diffeomorphism given by the left action of $g \in G$, then $\tau(g) = 0$. On the other hand, if G is connected, drawing g back to e along a path from e to g produces a smooth homotopy between $g : G \to G$ and id_G. Since Lefschetz number is a homotopy invariant, we find that $0 = \tau(g) = \tau(\mathrm{id}_G) = \chi(G)$.

Problem. (F15:02) Let $f, g : M \to N$ be smooth maps between smooth manifolds that are smoothly homotopic. Prove that if ω is a closed form on N, then $f^*\omega$ and $g^*\omega$ are cohomologous.

Solution.
We follow the presentation in Morita. We first demonstrate an easier proposition:

Proposition 1. *Let M be a smooth manifold, $\pi : M \times \mathbb{R} \to M$ the projection onto the first factor, and $i : M \to M \times \mathbb{R} : p \mapsto (p,0)$. Consider the induced maps*

$$\pi^* : H^*_{dR}(M) \to H^*_{dR}(M \times \mathbb{R})$$

and

$$i^* : H_{dR}(M \times \mathbb{R}) \to H^*_{dR}(M).$$

Then there exists a cochain homotopy

$$\Phi : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$$

between id *and $\pi^* i^*$: that is,*

$$\mathrm{id} - \pi^* i^* = d\Phi + \Phi d \quad \text{on } \Omega^k(M \times \mathbb{R}).$$

In particular, π^* is an isomorphism with inverse i^*.

Proof. Given an arbitrary k-form $\omega \in \Omega^k(M \times \mathbb{R})$, take local coordinates and write

$$\omega = \sum_I a_I(x,t) dx^I + \sum_J b_J(x,t) dt \wedge dx^J$$

where the first sum is over increasing multi-indices I of length k and the second is over increasing multi-indices J of length $k-1$. Define Φ by

$$\Phi\omega = \sum_J \left(\int_0^t b_J(x,s)\, ds \right) dx^J.$$

We claim that

$$d(\Phi\omega) + \Phi(d\omega) = \omega - \pi^* i^* \omega.$$

By linearity, we may separate into the following cases: $\omega = a(x,t)dx^I$, and $\omega = b(x,t)dt \wedge dx^J$. In the former case, $\Phi\omega = 0$ and

$$\begin{aligned} \Phi(d\omega) &= \left(\int_0^t \frac{\partial a}{\partial s}\, ds \right) dx^I \\ &= (a(x,t) - a(x,0))dx^I \\ &= \omega - \pi^* i^* \omega. \end{aligned}$$

In the latter case, $i^*\omega = 0$, so $(\mathrm{id} - \pi^* i^*)\omega = \omega$, and we compute:

$$\begin{aligned} d(\Phi\omega) &= d\left(\int_0^t b(x,s)\, ds\right) dx^J \\ &= \omega + \sum_{m=1}^n \left(\int_0^t \frac{\partial b}{\partial x^m}\, ds\right) dx^m \wedge dx^J, \end{aligned}$$

$$\begin{aligned} \Phi(d\omega) &= \Phi\left(-\sum_{m=1}^n \frac{\partial b}{\partial x^m} dt \wedge dx^m \wedge dx^J\right) \\ &= -\sum_{m=1}^n \left(\int_0^t \frac{\partial b}{\partial x^m}\, ds\right) dx^m \wedge dx^J. \end{aligned}$$

Therefore $d(\Phi\omega) + \Phi(d\omega) = \omega$, and the claim is proved in local coordinates. But the definition of Φ is independent of the choice of coordinates, since it only affects the real coordinate. We have thus shown that id and $i \circ \pi$ induce chain homotopic maps on the de Rham complex of $M \times \mathbb{R}$. It is now standard homological algebra that π^* is an isomorphism on cohomology with i^* as its inverse. $\square$

Lastly, we need to show that $f^* = g^*$ on cohomology. Let $F : M \times \mathbb{R} \to N$ denote the homotopy between f and g, such that $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$. Let $i_0, i_1 : M \to M \times \mathbb{R}$ be given by $i_0(p) = (p, 0)$ and $i_1(p) = (p, 1)$. Then clearly $f = F \circ i_0$ and $g = F \circ i_1$. The proof of the previous proposition shows that $i_0^* = i_1^* = (\pi^*)^{-1}$. Therefore

$$f^* = (F \circ i_0)^* = i_0^* F^* = i_1^* F^* = (F \circ i_1)^* = g^*.$$

Problem. (F15:03) For two smooth vector fields X, Y on a smooth manifold M, prove the formula

$$[L_X, i_Y]\omega = i_{[X,Y]}\omega,$$

where L_X is the Lie derivative in the direction of X, i_X is the interior product of X, and ω is a k-form for $k \geq 1$.

Solution. There are no tricks to this problem: it is a matter of computing exactly from the definitions. Recall the coordinate-independent formulas for Lie derivatives and interior products: if θ is a k-form, then

$$L_X\theta(Z_1, \ldots, Z_k) = X(\theta(Z_1, \ldots, Z_k)) - \sum_{i=1}^k \theta([X, Z_i], Z_1, \ldots, \widehat{Z_i}, \ldots, Z_k)$$

and

$$i_Y\theta(Z_1,\dots,Z_{k-1}) = \theta(Y,Z_1,\dots,Z_{k-1}).$$

We compute

$$i_{[X,Y]}\omega(Z_1,\dots,Z_{k-1}) = \omega([X,Y],Z_1,\dots,Z_{k-1}),$$

and

$$[L_X,i_Y]\omega(Z_1,\dots,Z_{k-1}) = L_X(i_Y\omega)(Z_1,\dots,Z_{k-1}) - i_Y(L_X\omega)(Z_1,\dots,Z_{k-1})$$

where

$$\begin{aligned} L_X(i_Y\omega)(Z_1,\dots,Z_{k-1}) &= X(i_Y\omega(Z_1,\dots,Z_{k-1})) - \sum_{i=1}^{k-1}(i_Y\omega)(Z_1,\dots,[X,Z_i],\dots,Z_{k-1}) \\ &= X(\omega(Y,Z_1,\dots,Z_{k-1})) - \sum_{i=1}^{k-1}\omega(Y,Z_1,\dots,[X,Z_i],\dots,Z_{k-1}) \end{aligned}$$

and

$$\begin{aligned} i_Y(L_X\omega)(Z_1,\dots,Z_{k-1}) &= (L_X\omega)(Y,Z_1,\dots,Z_{k-1}) \\ &= X(\omega(Y,Z_1,\dots,Z_{k-1})) - \omega([X,Y],Z_1,\dots,Z_{k-1}) \\ &\quad - \sum_{i=1}^{k-1}\omega(Y,Z_1,\dots,[X,Z_i],\dots,Z_k). \end{aligned}$$

Taking the difference now yields the identity.

Problem. (F15:04) Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be the 3-torus and $C = \pi(L)$, where $L \subset \mathbb{R}^3$ is the oriented line segment from $(0,1,1)$ to $(1,3,5)$ and $\pi : \mathbb{R}^3 \to M$ is the quotient map. Find a differential form on M which represents the Poincaré dual of C.

Solution. L defines a cycle, hence an element of $H_1(M;\mathbb{R})$. Its Poincaré dual is the unique element $[\eta]$ of $H^2_{dR}(M)$ such that the Poincaré dual map

$$PD([\eta])(\cdot) : H^1_{dR}(M) \to \mathbb{R} : [\theta] \mapsto \int_M \eta\wedge\theta$$

coincides with the dual of $[L]$ through the map induced by the pairing

$$([L],\cdot) : H^1_{dR}(M) \to \mathbb{R} : [\theta] \mapsto ([L],[\theta]) = \int_L \theta.$$

That is, we seek a 2-form η such that

$$\int_L \theta = \int_M \eta \wedge \theta$$

for all 1-forms θ. L is a loop on $\mathbb{T}^3$ that wraps in the x-direction once, in the y-direction 2 times, and in the z-direction 4 times. The de Rham cohomology of T^3 is generated as the exterior algebra over the coordinate 1-forms dx, dy, dz. From our description of L, we then have that

$$\int_L (adx + bdy + cdz) = a + 2b + 4c.$$

On the other hand, η is a 2-form, and can therefore be written as $\eta = Adx \wedge dy + Bdx \wedge dz + Cdy \wedge dz$. Then $\int_M \eta \wedge \theta$ is

$$\begin{aligned}\int_M (Adx \wedge dy + Bdx \wedge dz + Cdy \wedge dz) \wedge (adx + bdy + cdz) &= \int_M (Ac - Bb + Ca) dx \wedge dy \wedge dz \\ &= (Ac - Bb + Ca) \int_M dx \wedge dy \wedge dz \\ &= Ac - Bb + Ca.\end{aligned}$$

Therefore we must solve

$$a + 2b + 4c = Ac - Bb + Ca,$$

for which we can choose $C = 1$, $A = 4$, and $B = -2$ to obtain

$$\eta = 4dx \wedge dy - 2dx \wedge dz + dy \wedge dz.$$

Problem. (F15:05) Recall that the Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

and $S^2 = \mathbb{CP}^1$ with homogeneous coordinates $[z_1, z_2]$, then $\pi(z_1, z_2) = [z_1, z_2]$. Show that π does not admit a section, i.e. there exists no smooth map $s : S^2 \to S^3$ with $\pi \circ s = \mathrm{id}_{S^2}$.

Solution. This problem likely wins the prize for "largest ratio of irrelevant to relevant information in a qual problem;" I suspect this is intentional, designed to throw people off. It is enough to note that $H_2(S^2) = \mathbb{Z}$, while $H_2(S^3) = 0$. If such a map s were to exist, then the induced maps on homology would imply that $(\mathrm{id}_{S^2})_* = \pi_* s_*$. But $(\mathrm{id}_{S_2})_*$ on second homology is $\mathrm{id}_{\mathbb{Z}}$, while π_* and s_* are both zero, contradiction.

Problem. (F15:07) Show there is no smooth degree one map from $S^2 \times S^2$ to $\mathbb{CP}^2$.

Solution. First, let us determine the integer cohomology of $S^2 \times S^2$. Giving one copy of S^2 the standard cell structure with one 0-cell v and one 2-cell E, and the other copy the structure with one 0-cell w and one 2-cell F, the cellular chain complex is generated by:

i. One 0-cell $v \times w$, with boundary $\partial(v \times w) = 0$;

ii. Two 2-cells $v \times F$, $E \times v$ with boundaries $\partial(v \times F) = \partial(E \times w) = 0$;

iii. One 4-cell $E \times F$ with boundary $\partial(E \times F) = 0$.

Therefore

$$H^n(S^2 \times S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 4, \\ \mathbb{Z}^2 & n = 2, \\ 0 & n = 1, 3. \end{cases}$$

On the other hand the cohomology of $\mathbb{CP}^2$ is

$$H^n(\mathbb{CP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 2, 4, \\ 0 & n = 1, 3. \end{cases}$$

Now, let $f : S^2 \times S^2 \to \mathbb{CP}^2$ be a smooth map. Then f induces maps on cohomology

$$f^* : H^*(\mathbb{CP}^2; \mathbb{Z}) \to H^*(S^2 \times S^2; \mathbb{Z}).$$

There is an element $\omega \in H^2(\mathbb{CP}^2; \mathbb{Z})$ such that ω^2 (by which we are referring to the cup product structure) generates; $H^4(\mathbb{CP}^2)$, and naturally

$$\deg f = f^*(\omega^2) = (f^*\omega)^2.$$

Since $H^2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}^2$, if we let α, β be generators then $\alpha\beta$ generates $H^4(S^2 \times S^2; \mathbb{Z})$ and there exist integers a, b with

$$f^*\omega = a\alpha + b\beta.$$

Then

$$(f^*\omega)^2 = 2ab\alpha\beta.$$

Therefore $\deg f = 2ab$, and in particular there is no choice of integers a and b that makes $\deg f = 1$.

Problem. (F15:08) Show that $\mathbb{CP}^{2n}$, $n \geq 1$, is not a covering space of any space other than itself.

Solution. In S11:09 we showed that the Lefschetz number of any map from $\mathbb{CP}^{2n}$ to itself is nonzero, and therefore every self-map of $\mathbb{CP}^{2n}$ has a fixed point. Now suppose $\pi : \mathbb{CP}^{2n} \to X$ is a covering space, and consider the group of deck transformations. Every non-identity deck transformation is a homeomorphism of $\mathbb{CP}^{2n}$, and because deck transformations act freely on the cover, every non-identity deck transformation has no fixed points. But then no non-identity deck transformation can exist. Therefore the deck transformation group is trivial, and $\mathbb{CP}^{2n} \to X$ is the trivial cover $\mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$.

Problem. (F15:09) Given a continuous map $f : X \to Y$ between topological spaces, define
$$C_f = ((X \times [0,1]) \coprod Y) / \sim$$
where $(x,1) \sim f(x)$ for all $x \in X$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$. Show that there is a long exact sequence

$$\cdots \longrightarrow H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \longrightarrow \tilde{H}_{i+1}(C_f) \longrightarrow H_i(X) \xrightarrow{f_*} \cdots$$

where f_* is the homomorphism induced by f.

Solution.
This is a special case of S15:09.

Problem. (F15:10) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.

(a) Give a CW decomposition of $\mathbb{RP}^n$ for $n \geq 1$.

(b) Use the CW decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$.

(c) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

Solution.

(a) The standard CW structure on $\mathbb{RP}^n$ is built inductively in the following way. For $n = 0$ we just have a point, $\mathbb{RP}^0$. For $n \geq 1$, we attach the n-cell e^n to the $(n-1)$-skeleton, which is homeomorphic to $\mathbb{RP}^{n-1}$, along the boundary $\partial e^n \cong S^{n-1}$ via the standard map $S^{n-1} \to \mathbb{RP}^{n-1}$. This is homeomorphic to the quotient of S^n in the following way: since the quotient map identifies antipodal points of S^n, after identifying the lower hemisphere to the upper hemisphere $S^n/\sim$ is equivalent to the upper hemisphere $\cong D^n$ with antipodal points of the boundary ∂D^n identified. Since $\partial D^n \cong S^{n-1}$, and applying the inductive hypothesis $S^{n-1}/\sim \cong \mathbb{RP}^{n-1}$, we see precisely that $D^n/\sim$ is precisely e^n attached to $\mathbb{RP}^{n-1}$ along the boundary via $S^{n-1} \to \mathbb{RP}^{n-1}$.

(b) Example 2.42 of Hatcher.

(c) The easiest solution is via the homological characterization of orientability for closed manifolds. $\mathbb{RP}^n$ is orientable whenever n is odd, since in this case $H_n(\mathbb{RP}^n) = \mathbb{Z}$. On the other hand, if n is even, then $H_n(\mathbb{RP}^n) = 0$, so in this case $\mathbb{RP}^n$ is nonorientable.

21 Spring 2016

Problem. (S16:01) Consider the set of all straight lines in $\mathbb{R}^2$ (not necessarily through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Solution. Call this set M. Such a line can always be expressed in the form

$$ax + by + c = 0,$$

where at least one of a and b is nonzero. The coefficients (a,b,c) are unique, up to a nonzero multiplicative scalar. Therefore the set of all straight lines can be described as the set

$$\{(a,b,c) \in \mathbb{R}^3 : (a,b) \neq (0,0)\}/\mathbb{R}^*,$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is the multiplicative group of nonzero real numbers, acting on $\mathbb{R}^3$ by $\lambda.(a,b,c) = \lambda(a,b,c) = (\lambda a, \lambda b, \lambda c)$. Equivalently, by normalizing the nonzero elements of $\mathbb{R}^3$, this as a set is

$$\{(a,b,c) \in S^2 \subset \mathbb{R}^3 : (a,b) \neq (0,0)\}/\{\pm 1\} = (S^2 \setminus \{N,S\})/\{\pm 1\},$$

where $N = (0,0,1)$ and $S = (0,0,-1)$ are the north and south poles of S^2. Thus as a quotient space, this has the structure of $\mathbb{RP}^2 = S^2/\{\pm 1\}$, minus the orbits of N and S, which is a single point in $\mathbb{RP}^2$. Since $\mathbb{RP}^2$ is a 2-manifold, and

$M = \mathbb{RP}^2 \setminus \{[N]\}$ is an open subset of $\mathbb{RP}^2$, this determines a smooth 2-manifold structure on M.

M is a connected surface. Orientability can be determined from classification of surfaces: a compact surface is nonorientable if and only if it results from the gluing of one or more Möbius bands to a sphere with holes along the boundary. M deformation retracts to a sphere with two holes, along one of which is attached a Möbius band (since $\mathbb{RP}^2$ is a sphere with one hole and a Möbius band attached to the hole); therefore it is nonorientable.

Problem. (S16:02) Let X and Y be submanifolds of $\mathbb{R}^n$. Show that for almost every $a \in \mathbb{R}$, the translate $X+a$ intersects Y transversely.

Solution. We are looking to show that for each $p \in (X+a) \cap Y$,

$$T_p(X+a) + T_pY = T_p\mathbb{R}^n.$$

Define the map

$$F : X \times \mathbb{R}^n \to \mathbb{R}^n : (x,a) \mapsto x+a.$$

We claim that $F \pitchfork Y$. If $F^{-1}(Y) = \varnothing$, then the claim holds trivially. Suppose now that $p \in F^{-1}(Y)$, i.e. $p+a \in Y$. The tangent space $T_p(X \times \mathbb{R}^n)$ is isomorphic to $T_pX \oplus T_p\mathbb{R}^n$, and

$$dF_p(T_pX \oplus T_p\mathbb{R}^n) = T_{p+a}\mathbb{R}^n.$$

So the transversality condition holds trivially. By the transversality theorem, it now follows that $f_a = F(\cdot, a)$ is transverse to Y for almost every a, which is the claim to be proved.

Problem. (S16:03) Consider the vector field $X(z) = z^{2016} + 2016z^{2015} + 2016$ on $\mathbb{C} = \mathbb{R}^2$, by which we mean identify $T_z\mathbb{C} = \mathbb{C}$ and write $X(z) = z^{2016} + 2016z^{2015} + 2016 \in T_z\mathbb{C}$. Compute the sum of the indices of X over its isolated zeros.

Solution. First, X only has isolated zeros, by standard facts about roots of complex polynomials. Around each zero z_0, X is locally of the form

$$X(z) = a(z-z_0)^m + H(z),$$

where m is the multiplicity of the root and $H(z)$ is nonvanishing in a neighborhood of z_0. The local index is therefore given by the multiplicity of the root. Since the multiplicities of the roots of a complex polyonimal sum to its degree, the sum of the local indices is 2016.

Problem. (S16:04) Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

Solution. Let us write $n = \dim M$. Consider two copies of M, say M_1 and M_2. We form a compact n-dimensional manifold without boundary $\tilde{M}$ by smoothly attaching M_1 to M_2 along their common boundary:

$$\tilde{M} = M_1 \cup_{\partial M} M_2.$$

We now apply Mayer-Vietoris to $\tilde{M}$. There exist open sets $U, V \subset \tilde{M}$ with U, V diffeomorphic to M and $U \cap V$ deformation retracting onto ∂M. (One could, for instance, embed into Euclidean space and invoke the tubular neighborhood theorem to make this precise.) Then the Mayer-Vietoris sequence gives us the exact sequence

$$\cdots \longrightarrow H^k_{dR}(\tilde{M}) \longrightarrow H^k_{dR}(M) \oplus H^k_{dR}(M) \longrightarrow H^k_{dR}(\partial M) \longrightarrow \cdots$$

Since the sequence is exact and terminates on both ends, taking alternating sums of dimensions gives us (by finite-dimensional linear algebra)

$$\chi(\tilde{M}) - 2\chi(M) + \chi(\partial M) = 0.$$

Now we note that since $\tilde{M}$ is a compact odd-dimensional manifold without boundary, $\chi(\tilde{M}) = 0$. Therefore we have

$$-2\chi(M) + \chi(\partial M) = 0,$$

and the claim is proved.

Problem. (S16:05) Let M be a compact oriented n-manifold with de Rham cohomology group $H^1_{dR}(M;\mathbb{R}) = 0$ and let T^n be the n-torus. For which integers k does there exist a smooth map $f : M \to T^n$ of degree k?

Solution. f induces a map on de Rham cohomology

$$f^* : H^*_{dR}(T^n) \to H^*_{dR}(M).$$

We recall that if $\theta_1, \ldots, \theta_n$ are cohomology classes generating $H^1_{dR}(T^n)$, then $H^*_{dR}(T^n)$ is the exterior algebra over $\theta_1, \ldots, \theta_n$. In particular, since $f^*\theta_i = 0$ for all $i = 1, \ldots, n$, we have

$$f^*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}) = f^*\theta_{i_1} \wedge \cdots \wedge f^*\theta_{i_k} = 0$$

for all $(i_1, \ldots, i_k)$, and therefore $f^* : H^k_{dR}(T^n) \to H^k_{dR}(M)$ is the zero homomorphism for all $k \geq 1$. In particular, $f^* : H^n_{dR}(T^n) \to H^n_{dR}(M)$ is the zero homomorphism, and therefore $\deg f = 0$.

Problem. (S16:06) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with coordinates $(x,y) \in \mathbb{R}^2$ and let $p \in T^2$.

(a) Compute the de Rham cohomology of the punctured torus $T^2 - \{p\}$.

(b) Is the volume form $dx \wedge dy$ exact on $T^2 - \{p\}$?

Solution.

(a) There is a quick solution from noting that the punctured torus deformation retracts to a wedge of two circles, but here we do a Mayer-Vietoris argument for practice. The de Rham cohomology of T^2 is

$$H^i_{dR}(T^2) = \begin{cases} \mathbb{R} & i = 0, \\ \mathbb{R}^2 & i = 1, \\ \mathbb{R} & i = 2. \end{cases}$$

Let U be a small neighborhood around $\{p\}$ such that U is diffeomorphic to a disc $\mathbb{D} \simeq \{q\}$, and let $V = T^2 - \{p\}$. Then $U \cup V = T^2$, and $U \cap V \simeq \mathbb{D} - \{p\} \simeq S^1$. The Mayer-Vietoris sequence for this pair is then

$$\begin{tikzcd} 0 \arrow[r] & H^0_{dR}(T^2) \arrow[r] & H^0_{dR}(\{q\}) \oplus H^0_{dR}(T^2 - \{p\}) \arrow[r] & H^0_{dR}(S^1) \arrow[dll] & \\ & H^1_{dR}(T^2) \arrow[r] & H^1_{dR}(\{q\}) \oplus H^1_{dR}(T^2 - \{p\}) \arrow[r] & H^1_{dR}(S^1) \arrow[dll] & \\ & H^2_{dR}(T^2) \arrow[r] & H^2_{dR}(\{q\}) \oplus H^2_{dR}(T^2 - \{p\}) \arrow[r] & H^2_{dR}(S^1) \arrow[r] & 0 \end{tikzcd}$$

$\{q\}$ has trivial de Rham cohomology, and S^1 has 0 2nd cohomology, and $\mathbb{R}$ for 0-th and first cohomology. Since $T^2 - \{p\}$ is connected, $H^0_{dR}(T^2 - \{p\}) = \mathbb{R}$. Since $T^2 - \{p\}$ is orientable and noncompact, it has zero top cohomology. Therefore the sequence above is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R} \arrow[dll, rounded corners] & \\
 & \mathbb{R}^2 \arrow[r] & H^1_{dR}(T^2 - \{p\}) \arrow[r] & \mathbb{R} \arrow[dll, rounded corners] & \\
 & \mathbb{R} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since the alternating sum of dimensions of an exact sequence of finite-dimensional real vector spaces is zero, we then conclude that

$$1 - 2 + 1 - 2 + \dim H^1_{dR}(T^2 - \{p\}) - 1 + 1 = 0.$$

Thus $\dim H^1_{dR}(T^2 - \{p\}) = 2$, and

$$H^i_{dR}(T^2 - \{p\}) \cong \begin{cases} \mathbb{R} & i = 0, \\ \mathbb{R}^2 & i = 1, \\ 0 & i = 2. \end{cases}$$

(b) Yes. Since 2 is the top dimension, every 2-form is closed, so ω is a closed form. Because $H^2_{dR}(T^2 - \{p\}) = 0$, all closed 2-forms are exact. Therefore ω is exact.

Problem. (S16:07) Exhibit a space whose fundamental group is $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. Exhibit another space whose fundamental group is $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Solution. Let X_k be the space obtained by attaching a 2-cell to S^1 as follows: regarding $D^2 \subset \mathbb{C}$ as the unit disc in the complex plane, and S^1 as the unit circle, attach D^2 to S^1 along ∂D^2 via the map $z \mapsto z^k$. (Or, if you like, give S^1 the usual cell structure with 1 vertex and 1 oriented edge labeled a, and attach D^2 to S^1 along the word a^k.) Then $\pi_1(X_k) = \mathbb{Z}/k\mathbb{Z}$. Then $X_m \vee X_n$ witnesses the first group, $X_m \times X_n$ the second.

Problem. (S16:08) Let L_x be the x-axis, L_y the y-axis, L_z the z-axis in $\mathbb{R}^3$. Compute

$$\pi_1(\mathbb{R}^3 - L_x - L_y - L_z).$$

Solution. This space deformation retracts onto S^2 minus 6 points, which by stereographic projection maps to $\mathbb{R}^2$ minus 5 points, which deformation retracts onto a wedge of five circles. Therefore

$$\pi_1(\mathbb{R}^3 - L_x - L_y - L_z) \cong \pi_1(\bigvee_{i=1}^{5} S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

Problem. (S16:09) Let X be a topological space and $p \in X$. The reduced suspension ΣX of X is the topological space obtained from $X \times [0,1]$ by contracting $X \times \{0,1\} \cup \{p\} \times [0,1]$ to a point. Describe the relation between the homology of X and ΣX.

Solution. We describe the reduced suspension in another way. First, we quotient $X \times I$ by only collapsing $\{p\} \times I$ to a point. The resulting space, say $\tilde{X}$, deformation retracts onto X by drawing $\{x\} \times I$ down to $\{x\} \times \{0\}$ for $x \neq p$. Therefore $\tilde{X}$ has the same homology as X. We now take the suspension $S\tilde{X}$ of $\tilde{X}$ (see S14:10) to obtain the reduced suspension: $S\tilde{X} = \Sigma X$. Using the result of S14:10, we obtain

$$H_n(\Sigma X) \cong H_n(S\tilde{X}) \cong H_{n-1}(\tilde{X}) \cong H_{n-1}(X).$$

Problem. (S16:10) Consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3.$$

Let S^3 be the unit sphere in $\mathbb{R}^4$ and let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map.

(a) Evaluate

$$\int_{S^3} \iota^* \alpha.$$

(b) Let γ be the 3-form on $\mathbb{R}^4 \setminus \{0\}$ given by

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k},$$

$k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

Solution.

(a) Since ι is inclusion, ι^* is restriction. We observe that

$$d\alpha = 4dx_1 \wedge dx_2 \wedge dx_3 \wedge dx^4.$$

Let B^4 denote the unit ball in $\mathbb{R}^4$. Then $\partial B^4 = S^3$. By Stokes' theorem,

$$\int_{S^3} \iota^*\alpha = \int_{B^4} d\iota^*\alpha = \int_{B^4} \iota^* d\alpha = 4\,\mathrm{Vol}(B^4).$$

(b) If γ is closed and $f = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-k}$, then

$$\begin{aligned} d\gamma &= df \wedge \alpha + f \wedge d\alpha \\ &= -2k(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-k} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &\quad + 4(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-k} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

Therefore we must have $k = 2$. If γ is exact, then γ must also be closed, so γ can only be exact if $k = 2$. But because $\gamma = \alpha$ on S^3, and $\int_{S^3} i^*\alpha \neq 0$, it follows that γ is not exact even for $k = 2$, so γ is not exact for any $k \in \mathbb{R}$.

22 Fall 2016

Problem. (F16:01) Let M be a smooth manifold. Prove that for any two disjoint closed subsets A and B of M, there exists a smooth function $f : M \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

Solution.
For each $x \in B$, let U_x be an open coordinate neighborhood of x that does not intersect A; this can be done by taking any coordinate neighborhood and taking the intersection with $M \setminus A$. Then the collection $\{U_x : x \in B\} \cup \{M \setminus B\}$ is an open cover of M. Let $\{\psi_x\} \cup \{\psi\}$ be a smooth partition of unity subordinate to this cover: that is, supp $\psi_x \subset U_x$, supp $\psi \subset M \setminus B$, and $\psi + \sum_{x \in B} \psi_x \equiv 1$. Set $f = \sum_{x \in B} \psi_x$. Then $f = 1$ on B, because ψ is supported outside of B and thus $\psi(x) = 0$ if $x \in B$. Also, $f = 0$ on A, because for all $x \in B$ ψ_x is supported outside of A.

Problem. (F16:02) Let $M \subset \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

Solution.
We first show that if $N > 2k+1$ and M is immersed into $\mathbb{R}^N$, then M can be immersed into $\mathbb{R}^{N-1}$. This proof is found in Lee.

Consider the following family of linear maps: for $v \in \mathbb{R}^N \setminus \{0\}$, let $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the projection onto the orthogonal complement of the span of v, which we identify with $\mathbb{R}^{N-1}$ up to Euclidean isometry. We claim that for almost every $v \in \mathbb{R}^N \setminus \{0\}$, the restriction $\pi_v|_M$ is an injective immersion $M \to \mathbb{R}^{N-1}$.

$\pi_v|_M$ is injective if for all distinct $p,q \in M$, $p-q$ is not parallel to v. Noting that π_v is linear, and therefore the same as its differential, $\pi_v|_M$ is an immersion if for all $p \in M$, $T_pM \subset \mathbb{R}^N$ does not contain any nonzero vectors parallel to v. Let $\Delta_M = \{(p,p) : p \in M\}$ be the diagonal of M, and let $M_0 = \{(p,0) \in TM : p \in M\}$. Define the following maps:

$$\kappa : M \times M \setminus \Delta_M \to \mathbb{RP}^{N-1} : (p,q) \mapsto [p-q],$$

$$\tau : TM \setminus M_0 \to \mathbb{RP}^{N-1} : (p,w) \mapsto [w],$$

where for $u \in \mathbb{R}^N \setminus \{0\}$ we denote the equivalence class of u in $\mathbb{RP}^{N-1}$ by $[u]$. These are smooth maps (because they are compositions of smooth maps), and $\pi_v|_M$ is an injective immersion if and only if $[v]$ is not in the images of κ and τ. κ and τ both have domains of dimension $2k < N-1$, and map into an $(N-1)$-dimensional manifold; therefore im κ and im τ have measure zero in $\mathbb{RP}^{N-1}$. Therefore the set of $v \in \mathbb{R}^N \setminus \{0\}$ so that $\pi_v|_M$ is an injective immersion is dense. Choosing such a v, we obtain our desired immersion of M into $\mathbb{R}^{N-1}$.

Iteration now allows us to assume M is injectively immersed into $\mathbb{R}^{2k+1}$. Now we show that this can be upgraded to an immersion (not necessarily injective) into $\mathbb{R}^{2k}$. Give M the Riemannian structure induced by the Euclidean inner product. Let $|w|$ denote the length of a tangent vector $w \in T_pM$. Define the unit tangent bundle $UM \subset TM$:

$$UM = \{(p,w) \in TM : |w| = 1\}.$$

Now define

$$\sigma : UM \to \mathbb{RP}^{2k} : (p,w) \mapsto [w].$$

Then σ is a smooth map taking UM (which is of dimension $2k-1$) to $\mathbb{RP}^{2k}$ (of dimension $2k$). Therefore by Sard's theorem, im σ has measure zero in $\mathbb{RP}^{2k}$, and thus we may find $[v] \in \mathbb{RP}^{2k}$ that is not in im σ. Then naturally $[v]$ is also not in im τ, where $\tau : TM \setminus M_0 \to \mathbb{RP}^{2k}$ is defined as above, since the map into projective space depends only on the direction of w, not its length. So we obtain as before an immersion $M \to \mathbb{R}^{2k}$.

Problem. (F16:03) Let $U_1, \ldots, U_n$ be n bounded, connected, open subsets of $\mathbb{R}^n$. Prove there exists an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ that bisects every U_i: that is, if A and B are the two half-spaces that form $\mathbb{R}^n \setminus H$, then

$$\text{volume}(U_i \cap A) = \text{volume}(U_i \cap B)$$

for all $i = 1, \ldots, n$.

Solution.
(Remark: This is known as the ham sandwich theorem.)
Define a map $g : S^{n-1} \to \mathbb{R}^{n-1}$ as follows. A point $v \in S^{n-1}$ and $s \in \mathbb{R}$ determines a unique hyperplane $H(s,v)$ with normal vector v and through the point sv. Let $A(s,v)$ be the half-space in on the side of $H(s,v)$ toward which v points, and let $B(s,v)$ be the opposing half-space. Then define $f = (f_1, \ldots, f_n)$ by

$$f_i(s,v) = \text{volume}(U_i \cap A(s,v)) - \text{volume}(U_i \cap B(s,v)).$$

We note that for fixed v, there exists at least one $s \in \mathbb{R}$ so that $f_n(s,v) = 0$: this follows from the intermediate value theorem. If there is more than one such s, then the set of all such s lie in a compact interval $[a,b] \subset \mathbb{R}$, because U_n is compact and therefore lies completely on one side of $H(s,v)$ for sufficiently positive or sufficiently negative s. Therefore we can pick a canonical hyperplane $H(s,v)$ for which $f_n(s,v) = 0$ by taking $s = a$, the most negative value for which $H(s,v)$ bisects U_n. Now define $g = (g_1, \ldots, g_{n-1})$ by

$$g_i(v) = f_i(s,v)$$

where s is the canonical value chosen above. Then g is a continuous map $S^{n-1} \to \mathbb{R}^{n-1}$, and note moreover that $g(-v) = -g(v)$. By the Borsuk-Ulam theorem, $g(v) = 0$ for some v, which implies that the required hyperplane exists.

Problem. (F16:04) Show that

$$D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - x_4 dx_2) \subset T\mathbb{R}^4$$

is a smooth distribution of rank 2, and determine whether D is integrable.

Solution. To show D is a smooth distribution of rank 2, it suffices to show that $\omega = dx_3 - x_1 dx_2$ and $\theta = dx_1 - x_4 dx_2$ are everywhere linearly independent one-forms; this is obvious since dx_3 and dx_1 are everywhere independent. We claim that D is not integrable. It is equivalent to show that there cannot exist 1-forms $\eta_1, \eta_2, \eta_3, \eta_4$ such that

$$d\omega = \eta_1 \wedge \omega + \eta_2 \wedge \theta,$$
$$d\theta = \eta_3 \wedge \omega + \eta_4 \wedge \theta.$$

(That is, the integrability conditions for the system of Pfaffian equations $\omega = \eta = 0$ on D cannot be satisfied.) We compute

$$d\omega = -dx_1 \wedge dx_2,\ d\theta = -dx_4 \wedge dx_2$$

and thus the integrability conditions require us to solve

$$\begin{aligned} -dx_1 \wedge dx_2 &= \eta_1 \wedge (dx_3 - x_1 dx_2) + \eta_2 (dx_1 - x_4 dx_2) \\ &= \eta_2 \wedge dx_1 + (-x_1\eta_1 - x_4\eta_2) \wedge dx_2 + \eta_1 \wedge dx_3, \end{aligned}$$

$$\begin{aligned} -dx_4 \wedge dx_2 &= \eta_3 \wedge (dx_3 - x_1 dx_2) + \eta_4 (dx_1 - x_4 dx_2) \\ &= \eta_4 \wedge dx_1 + (-x_1\eta_3 - x_4\eta_4) \wedge dx_2 + \eta_3 \wedge dx_3. \end{aligned}$$

We now observe that no choice of η_j allows us to solve the second equation. For this amounts to solving the system

$$\begin{aligned} 0 &= \eta_4 \wedge dx_1, \\ -dx_4 \wedge dx_2 &= (-x_1\eta_3 - x_4\eta_4) \wedge dx_2, \\ 0 &= \eta_3 \wedge dx_3. \end{aligned}$$

Then η_3 and dx_3 must be linearly dependent, i.e. $\eta_3 = f dx_3$, and similarly $\eta_4 = g dx_1$. But then

$$(-x_1\eta_3 - x_4\eta_4) \wedge dx_2 = -x_1 f dx_3 \wedge dx_2 - x_4 g dx_1 \wedge dx_2,$$

and there is no hope of choosing f or g such that this is equal to $-dx_4 \wedge dx_2$ since the $dx_i \wedge dx_j$ form a basis of the space of 2-forms.

Problem. (F16:07) Let X be a connected CW complex with finite fundamental group. Show that any map $F : X \to (S^1)^n$ is nullhomotopic.

Solution. See also F11:03. Since $\pi_1((S^1)^n) = \mathbb{Z}^n$, which has no finite subgroups, it follows that $F_* : \pi_1(X) \to \pi_1((S^1)^n)$ is trivial. Therefore F lifts to a map to the universal cover, $\tilde{F} : X \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, $\tilde{F}$ is nullhomotopic, and therefore $F = p \circ \tilde{F}$ is nullhomotopic (here $p : \mathbb{R}^n \to (S^1)^n$ is the covering map).